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Spectral sequences and parametric normal forms

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ABSTRACT

This paper extends recent developments on spectral sequence approach on the simplest normal form theory to parametric cases. In this paper, the method is first applied to two simple examples (parametric single zero and parametric resonant saddle singularities) and then, to parametric generalized Hopf singularity. We provide the simplest parametric normal form for these cases.

1. Introduction

This paper extends the recent developments of normal form theory (without parameters) via the spectral sequence method to parametric normal forms. We consider the following system:

$$\dot{x} = v(x, \mu), \tag{1.1}$$

where v represents a formal vector field, $x = (x_1, x_2, \ldots, x_p) \in \mathbb{F}^p$, $\mu = (\mu_1, \mu_2, \ldots, \mu_q) \in \mathbb{R}^q$, $v(0, \mu) = 0$. We refer to x_i as a state variable and μ_i as a parameter.

The idea of normal form theory is to simplify system (1.1) via near-identity change of state variables (near-identity ensures that such transformations are locally invertible) such that certain local dynamical features (such as stability of equilibrium, limit cycles, etc.) of the system do not change.

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Then, instead of the original vector field, one analyzes the normalized vector field, see, e.g., [10,16,29,34,37]. Real life problems modeled by scientists and engineers usually involve parameters, and obtaining the transformations between the original system and its normalized system is fundamentally important in applications. Thus, computing these transformations is one of the main goals of parametric normal form theory, see, e.g., [38,39]. Since μ represents parameters such as control, perturbation or bifurcation parameters, we assume that parameters are constants with respect to time and are in the neighborhood of the origin. The later indicates that division by parameters are not permitted. In order to effectively simplify systems with parameters such that their stability and bifurcation analysis would be possible, we may also need to use time rescaling and reparametrization, see, e.g., [1,2,30–32].

When time rescaling and changes of state variables are used, we call the simplified system *orbital normal form*. Note that some features, that usually remain the same between the original systems and their normal forms, may be changed when time rescaling is used. For example, a system and one of its orbitally equivalent systems can have cycles that look like the same closed curves in the phase space but can have different periods. When the change of state variable, time rescaling and reparametrization are used, we call the simplified system *parametric normal form*.

The near-identity changes of state variables form a group under Campbell–Hausdorff–Baker formula as it does for near-identity time rescaling and near-identity reparametrization under multiplication of functions and compositions of functions, respectively. Linear changes of variables also form a group. They all generate a subgroup in the group of filtration-preserving automorphisms of the vector fields (they act on vector fields like group acting on a vector space). We may refer to this subgroup of automorphisms as the *group of changes of variables* (variables are time, parameters and state variables). Given this, parametric normal form theory is to provide a *unique* representative for any orbit generated by this group associated with a given parametric system. Linear changes of state variables (i.e., only depending on state variables) are used to put the linear part of the system in a certain form. We assume this is already done and thus, it shall not be used in our normal form computations. Linear reparametrization (independent of state variables) is necessary for a possible reduction of number of parameters that explicitly appear in the system.

The spectral sequence method is one of the most elegant and powerful methods of computation. It has been applied in different branches of mathematics and has helped in solving many difficult computational problems. Arnold [3,4] was the first to apply this method on singularity theory, and recently the method has been further elaborated in normal form theory [9,22,27,28]. Sanders [27,28] and Murdock [22] mainly focused on normal forms of non-parametric vector fields while Benderesky and Churchill [9] applied the method to matrix normal forms. The later put their results in an innovative general setting, by introducing a group structure acting on a vector space. We use their results to establish a foundation for parametric normal forms of vector fields which also includes reparametrization as well as time rescaling. The existing results in [9,22,27,28] use neither time rescaling nor reparametrization while the use of these is the main new feature of this paper. Obviously, our method is also suitable for obtaining the spectral sequences for the infinite level orbital normal forms of non-parametric vector fields.

Since the spectral sequence spaces appear as quotient spaces, we require a rule on how to choose a representative from each coset. We imply this kind of rule from what is commonly defined as *style* in the literature. A style is a rule on how to choose a unique complement space of a subspace. Representatives of the infinite level quotient spaces in the spectral sequences are our candidates for the most simplified normal forms. This is because all the spectral data from the transformation space at the infinite level have been used to simplify the system. Different names are used in the literature such as *unique normal form, hypernormal form, infinite order normal form, infinite level normal form and the simplest normal form.* We follow Murdock and Sanders [22] to use the infinite *level* normal forms, for more details see [21].

Spectral sequences have been introduced for normal form calculations in [9,22,27,28]. But even in these papers, the spectral sequence method is usually kept in the background as a guide to the calculations, while the calculations themselves are done in the ordinary way [22]. Here we will show how it would look like to employ the notations of the spectral sequences in full. This has the advantage that the exact range of non-uniqueness is displayed all times in every step. The associated disadvantage is the complexity of the notation. It seems worthwhile to have some examples of this type of calculation in the literature, even if most users may choose to use a simpler notation and do some extra work on the side to keep track of the range of non-uniqueness when that is desired. In other words, the method of spectral sequences provides a systematic way for computing the spectral data left at our disposal to carry them forward for further simplification. We believe that the spectral sequence method is powerful and seems necessary for the cases in which time rescaling is used. Indeed, the computation of the infinite level orbital normal form requires a systematic algebraic approach to track the spectral data, see however [30–32] where a non-algebraic method is used to classify orbital normal forms for germs of analytic complex vector fields. Spectral sequence method determines this and demonstrates how to compute the infinite level orbital (and parametric) normal forms. This information for non-orbital (non-parametric) normal forms has been computed in the literature by alternative approaches, e.g., using multiple Lie brackets in the homological equations by defining higher level normal forms. Indeed, in this way one computes the spectral data at each step. This approach cannot be easily applied for time rescaling, since the ring associated with time rescaling is a torsion free ring acting on the space of vector fields as a torsion free module structure [13,14]. Besides, some time rescaling terms are often needed for further simplification of the system in higher levels. Therefore, the notion of higher level maps for time rescaling transformations cannot be applied unless through a well-coordinated method with other transformations of state and parameters.

The remaining of this paper is organized as follows. Section 2 describes the relation between unique complement spaces and quotient spaces based on notions of style and costyle. Section 3 describes the group of changes of variables, their correlations and how they act on a system. Note that the results presented in Sections 2 and 3 may seem to be unrelated, while all these pieces have been introduced to work together to establish our results in the following sections. The cohomology spectral sequences and how to use it for computing parametric normal forms are briefly discussed in Section 4. Then, the method is applied on three main examples, each presented in a separate section. In Section 5, we show that under certain technical conditions, a perturbation of the scalar ($x \in \mathbb{F}$) differential equation (single zero singularity)

$$\dot{x} = \sum_{i=k}^{\infty} a_i x^i$$

can be put into infinite level parametric normal form

$$\dot{x} = x^k + \sum_{i=1}^{k-1} x^i \mu_i,$$

see also [36] for the simplest orbital normal form of zero singularity with a single parameter. A perturbation of the planar ($x, y \in \mathbb{F}$) resonant saddle singularity system

$$\begin{cases} \dot{x} = mx + \text{h.o.t.}, \\ \dot{y} = -ny + \text{h.o.t.}, \end{cases} \quad \text{for } m, n \in \mathbb{N}, \text{ h.o.t. stands for nonlinear terms,}$$

is considered in Section 6. Assuming some technical conditions, we prove that its infinite level parametric normal form is

$$\begin{cases} \dot{x} = mx, \\ \dot{y} = -ny + ax^{kn}y^{km+1} + (b + \mu_{k+1})x^{2kn}y^{2km+1} + \sum_{i=0}^{k-1} x^{in}y^{im+1}\mu_{i+1}, \end{cases} \quad \text{for some } k \in \mathbb{N}.$$

Finally, Section 7 deals with a perturbation of the planar ($z, \bar{z} \in \mathbb{C}$) system (Hopf singularity)

$$\dot{z} = iz + \text{h.o.t.}, \quad \text{h.o.t. stands for nonlinear part in terms of } z, \bar{z},$$

where for simplicity the equation for $\dot{\bar{z}}$ has been ignored. Again some technical conditions are assumed and it is proved that the infinite level parametric normal form of this system is

$$\dot{z} = iz + \sum_{l=1}^{k} z^l \bar{z}^{l-1} \mu_l + z^{k+1} \bar{z}^k + (a + \mu_{k+1}) z^{2k+1} \bar{z}^{2k}, \quad \text{for some } k \in \mathbb{N},$$

see [14,15,38,39] for alternative parametric normal forms. Finally, conclusion is drawn in Section 8.

2. Quotient and complement spaces

In this section, we define the notions of formal basis $\mathscr{B}$, formal basis complement ($\mathscr{B}$-complement) of a subspace, and formal basis normal form style and costyle. We also describe how to use formal bases to identify quotient spaces with complement spaces (see also [13,14]).

Let $V = \prod_{i=1}^{\infty} V_i$, where V_i's are finite dimensional vector spaces over a field $\mathbb{F}$ of characteristic zero which in our applications will be $\mathbb{R}$ or $\mathbb{C}$. We call V a *locally finite graded vector space* and each V_i a *homogeneous space of grade* i. We denote by $J_k(v)$ the *k-jet* of the vector field v defined by $J_k(v) := \sum_{i=1}^{k} v_i$, where $v_i \in V_i$. In order to make the paper more readable, we follow Murdock and Sanders [21,24] to write $V = \bigoplus_{i=1}^{\infty} V_i$, indicating the elements of V, written as countable sums (i.e. formal series $v_1 + v_2 + \cdots$) rather than as sequences $(v_1, v_2, \ldots)$. This notation should not be confused with the common direct sum of vector spaces whose elements can only be represented as a finite sum of non-zero terms. A finite or countable *ordered* set (sequence) $\mathscr{B} = \{e_j \mid j \in \mathbb{N}\} \subseteq V$ is called a *formal basis* for V if every element $v \in V$ is uniquely represented by $v = \sum a_j e_j$ for $a_j \in \mathbb{F}$ where the sum is either finite or infinite.

A filtration is associated with any grading. Indeed, let $\mathscr{F}^k V = \{\sum_{i=k}^{\infty} v_i \mid v_i \in V_i\}$ and call $\mathscr{F} = \{\mathscr{F}^k V\}_{k=1}^{\infty}$ a filtration associated with the graded vector space $V = \bigoplus_{i=1}^{\infty} V_i$. The filtration $\mathscr{F}$ induces a topology on V by considering $\{v + \mathscr{F}^k V\}$ as a neighborhood of $v \in V$. The induced topology $\tau_{\mathscr{F}}$ from $\mathscr{F}$ is called *filtration topology*, see [5]. The filtration $\mathscr{F}$ is *Hausdorff*, i.e., $\bigcap_p \mathscr{F}^p V = \{0\}$, and *exhaustive*, i.e., $\mathscr{F}^0 V = V$.

For a vector subspace W of V, a unique complement space W^c is defined as the vector space span of a subset of a known formal basis $\mathscr{B}$ that provides a formal basis for W^c. Given W, we inductively choose the least natural number n_k such that e_{n_k} is not an element of $W \oplus \operatorname{span}\{e_{n_i} \mid i < k\}$. We continue this until either the process terminates for a finite number N (i.e., $W^c = \operatorname{span}\{e_{n_i} \mid i \leqslant N\}$) or we obtain an infinite sequence $\{e_{n_i}\}$ (i.e., $W^c = \operatorname{span}\{e_{n_i}\}$). From now on, we refer to W^c as the unique $\mathscr{B}$-complement space for W in V. Notice that the order of the basis elements $e_1, e_2, \ldots$ in $\mathscr{B}$ has a strong effect on the construction of W^c; if $i < j$ then e_i is *preferred* over e_j in choosing the basis for W^c. Observe that $W_1 \subset W_2$ implies $W_2^c \subset W_1^c$; this is because if e_m is not an element of $\operatorname{span}\{e_i \mid i < m\} + W_2$, then e_m is also not an element of $\operatorname{span}\{e_i \mid i < m\} + W_1$.

Note that W^c is the $\mathscr{B}$-complement space for W, but W is not always the $\mathscr{B}$-complement space for W^c. The only case in which W and W^c are $\mathscr{B}$-complement spaces for each other is when W is generated by the formal basis. Therefore, we call a space W *a* $\mathscr{B}$*-space* when $W = \operatorname{span}(\mathscr{B} \cap W)$. Since complement spaces are $\mathscr{B}$-subspaces in V, $W = (W^c)^c$ if and only if W is a $\mathscr{B}$-space. Furthermore, for any $\mathscr{B}$-space W we always have $W^c = \operatorname{span}(\mathscr{B} \cap (V \setminus W))$, where $V \setminus W$ represents the set of elements in V that are not in W. Any summation of two $\mathscr{B}$-subspaces is a $\mathscr{B}$-subspace.

A formal basis $\mathscr{B}$ defines a unique $\mathscr{B}$-projection π_W onto a subspace W of V. Recall that $V = W \oplus W^c$ and that given W, we have chosen a subset $\{e_i\}$ of $\mathscr{B}$ that forms a basis for W^c. Therefore, for any $v \in V$ there exist unique scalars $a_{n_1}, a_{n_2}, \ldots,$ and $w \in W$ such that $v = w + \sum a_{n_i} e_{n_i}$. Now define $\pi_W(v) = w$. As a simple example to illustrate this, consider $\pi_W(e_1 + 2e_2)$ for which $V = \mathbb{F}^2$ with standard ordered basis $\mathscr{B} = \{e_1, e_2\}$ and $W = \operatorname{span}\{e_1 + e_2\}$. Therefore, $W^c = \operatorname{span}\{e_1\}$ and $\pi_W(e_1 + 2e_2) = 2e_1 + 2e_2$.

The method of spectral sequences works with quotient spaces rather than complement spaces. The notion of $\mathscr{B}$-complement spaces helps to present a quotient space by unique representative of its cosets, see also [27]. Any coset $v+W$ can be written uniquely as $u+W$ for some $u \in W^c$. We often identify V/W with W^c by mapping $u+W \mapsto u$. To avoid confusions in this section and Lemma 4.1, any quotient space E/F is denoted for the set of cosets, and $\overline{E/F}$ denotes the $\mathscr{B}$-complement space of F in E. However, for convenience in the rest of this paper, we shall use E/F for both purposes. The formal basis $\mathscr{B}$ for V gives a natural formal basis for the quotient space V/W, i.e., $\mathscr{B}_{V/W} = \{e_{n_i} + W\}$ where $\mathscr{B} \cap W^c = \{e_{n_i}\}$. Thus, we have the following lemma.

Lemma 2.1. *Let W and $\mathscr{B}$ be a vector subspace and a formal basis for V, respectively. Then*:

(a) *For any quotient space V/W, there exists* a unique $\mathscr{B}$-space U *in V such that $\overline{V/W} = \overline{V/U}$.*
(b) *Let W_1 be a $\mathscr{B}$-subspace in V, and W_2 be a vector subspace of W_1. Then, $\overline{V/W_1} = \overline{(\overline{V/W_2})/(\overline{W_1/W_2})}$.*

Proof. Since $W^c = \overline{V/W}$, we define $U = (W^c)^c$. Then, $\overline{V/U} = U^c = ((W^c)^c)^c = W^c = \overline{V/W}$. This proves the existence of U. For uniqueness suppose U is $\mathscr{B}$-space and $\overline{V/U} = \overline{V/W}$. Then, $U = (U^c)^c = \overline{V/U}^c = (W^c)^c$. This completes the proof of (a).

For (b), W_1 needs to be a $\mathscr{B}$-space in order that $\overline{W_1/W_2}$ be a well-defined $\mathscr{B}$-complement space of W_2 in W_1. Note that $\mathscr{B} \cap W_2^c = (\mathscr{B} \cap W_1 \cap W_2^c) \cup (\mathscr{B} \cap W_1^c)$ and $(\mathscr{B} \cap W_1 \cap W_2^c) \cap (\mathscr{B} \cap W_1^c) = \emptyset$. Thereby, $\overline{W_1/W_2} = \operatorname{span}(\mathscr{B} \cap W_2^c \cap W_1)$ is a $\mathscr{B}$-space. Since $\overline{V/W_2} = W_2^c$ is also a $\mathscr{B}$-space, $\overline{(\overline{V/W_2})/(\overline{W_1/W_2})}$ is spanned by the elements of $\mathscr{B} \cap W_2^c$ that are not in $\mathscr{B} \cap W_2^c \cap W_1$. Thus, $\overline{(\overline{V/W_2})/(\overline{W_1/W_2})}$ is spanned by $\mathscr{B} \cap W_1^c$. This implies $\overline{(\overline{V/W_2})/(\overline{W_1/W_2})} = \overline{W_2^c/\overline{W_1/W_2}} = \overline{V/W_1}$. $\square$

Roughly speaking, a system is in normal form if it lies in a complement to a *removable space,* which is the image of some operator (a Lie bracket or sum of brackets, when only state change of variables are used). A style is a rule for choosing the complement of the removable space, see [21–23]. Examples are the inner product style and the *sl*(2) style. When such a rule is intended for the space of transformations rather than the space of vector fields, it is called *costyle* [23].

In this paper, our normal forms follow the formal basis style. Further, since vectors remaining in the complement spaces are the candidates for normal forms, our normal forms can be presented by terms of the system. Thereby, by choosing the order of the formal basis elements we automatically determine the approach in which terms will be eliminated. The *terms* succeeding the *others* in the *ordering* of the formal basis are in priority for elimination. This is, indeed, one of the main purposes of using formal bases in this paper. Despite other common styles, we believe it is an advantage for our approach to decide the priority in eliminating *certain terms* from the system well in advance of calculations by setting a fixed order on formal basis. For example, in the case of Hopf singularity, we practically give priority to the *amplitude* terms rather than *phase* terms of the same grade. This way we calculate the complement spaces based on the order of the formal basis rather than making a choice at each step of computations.

Remark 2.2. Formal basis and the inner product styles are styles that work for higher level normal forms, while the *sl*(2) style has not been extended from the first level normal form to higher levels. Another advantage for formal basis style is that other styles, e.g., the inner product and *sl*(2) styles, can be expressed as formal basis styles by making the correct choice of $\mathscr{B}$. A choice of an orthogonal formal basis (with respect to an inner product) unifies the formal basis and the inner product style normal form. In order to obtain an *sl*(2) normal form style for a nilpotent singularity via a formal basis style, we can choose a formal basis such that it is *sl*(2)-invariant. Thus, if the basis terms in the classical *sl*(2) style normal form are ordered before other basis terms, the first level normal form in formal basis style will coincide with classical *sl*(2) style normal form. Finally, it is imperative to distinguish the formal basis from the method of formal decompositions (described in [13,14]).

The order of formal basis is very important in obtaining normal forms. Throughout this paper we consider the following rules on every formal basis:

(I) Terms of lower grades precede terms of higher grades in the ordering.
(II) Terms without parameter precedes terms with parameter in the ordering when they have the same grade.
(III) For the case of transformation spaces, state terms precede time terms while parameter terms succeed time terms, when they all have the same grade. (Note that this is only relevant when costyle is concerned, see below.)

The reasons behind the rules (I) and (II) are easy to observe; the rule (I) means that lower grade terms are in priority for elimination while the rule (II) gives the priority to omit terms with parameters rather than terms without parameters of the same grade.

Remark 2.3. We may or may not require additional rules for each singularity. Nevertheless, the above rules (with possible additional rules) are not sufficient to set up a unique order for $\mathscr{B}$, but yet they are sufficient that any fixed order satisfying them would lead to a unique parametric normal form for each of the singularities considered in this paper; see Sections 5–7.

Any element of the formal basis multiplied with a scalar is called a *term*. We follow the common practice to denote $\mu^{\mathbf{m}} = \mu_1^{m_1}\mu_2^{m_2}\cdots\mu_p^{m_q}$ with $|\mathbf{m}| = m_1 + m_2 + \cdots + m_q$ for any $\mathbf{m} = (m_1, m_2, \ldots, m_q) \in \mathbb{N}^q$.

The notation π_W should be distinguished from $\pi_i(\mathbf{v})$; indeed, $\pi_i(\mathbf{v}) = v_i$ where v_i is the i-th component of the vector $\mathbf{v} \in \mathbb{F}^n$ for $1 \leqslant i \leqslant n$. Further, $\mathbf{e}_i \in \mathbb{F}^n$ denotes the i-th element of the standard basis, i.e., $\pi_j(\mathbf{e}_i) = \delta_{ij}$, where δ_{ij} stands for the Kronecker delta function and should not be confused with δ_α which represents grading functions. Further, we denote $\mathbf{v} = [v_1, v_2, \ldots, v_n]$ for a column vector and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ for a row vector.

3. Change of variables and their structures

In this section, we review the algebraic structures of invertible formal change of variables. We first present five different kind of changes of variables and then, the link between them is illustrated in terms of semidirect products of subgroups. We prove that any finite (infinite, if they are convergent with respect to filtration topology) composite of these can be described uniquely by a composite of five transformations.

Any near-identity change of state variable can be generated by a formal function $S : \mathbb{C}^p \times \mathbb{C}^q \to \mathbb{C}^p$ (containing only monomials that are either of at least degree 2 in x, or of at least degree 1 in x and at least degree 1 in μ) as follows. Denote $\mathscr{S}$ for the set of all such functions and let $x(t; y, \mu)$ be the solution of the initial value problem:

$$\frac{d}{dt}x(t; y, \mu) = S\big(x(t; y, \mu), \mu\big),$$
$$x(0, y, \mu) = y. \tag{3.1}$$

Now for any $S \in \mathscr{S}$ define the mapping $\phi_S : \mathbb{C}^p \times \mathbb{C}^q \to \mathbb{C}^p$ by $\phi_S(y, \mu) = x(1; y, \mu)$, where $x(1; y, \mu)$ is the solution of Eq. (3.1). This is a near-identity coordinate transformation given by $x = \phi_S(y, \mu)$, transforming the new variable y to the old variable x. This is the standard Lie transform in the sense of Hori (not Deprit), see [21, format 2b]. If we begin with the vector field v expressed in (x, μ) given by Eq. (1.1) and apply the coordinate change ϕ_S, we obtain the same vector field in the form $\Phi_S v$, which is now expressed in the variables (y, μ). Then,

$$(\Phi_S v)(y,\mu) = \big[(D\phi_S)(y,\mu)\big]^{-1} v\big(\phi_S(y,\mu),\mu\big)$$
$$= v + [v,S] + \frac{1}{2}\big[[v,S,],S\big] + \frac{1}{6}\big[\big[[v,S],S\big],S\big] + \cdots, \tag{3.2}$$

where D is the matrix of partial derivatives with respect to y, with μ held constant and Lie bracket is defined by $[u,w] = w'u - u'w$ (which equals the Wronskian(u,w), see, e.g., [21]). Thus, system (1.1) is transformed into $\dot{y} = (\Phi_S v)(y,\mu)$. This imposes a Lie algebraic structure on the space of all formal vector fields.

Near-identity reparametrization can also be generated by formal functions like $P : \mathbb{C}^q \to \mathbb{C}^q$, where P contains no constant or linear term. We denote the set of all such functions by $\mathscr{P}$. Then, for any $P \in \mathscr{P}$ a near-identity change of parameters is defined by

$$\mu = \psi_P(\nu) = \nu + P(\nu). \tag{3.3}$$

Similarly, denote $\Psi_P v$ for the updated vector field v in terms of new variable ν through the formula

$$(\Psi_P v)(x,\nu) = v\big(x,\psi_P(\nu)\big) = v\big(x,\nu+P(\nu)\big) = \sum_{n=0}^{\infty}\frac{1}{n!}D_\mu^n(v,P), \tag{3.4}$$

where $D_\mu^n(v,P)$ denotes the n-th order formal Fréchet derivative of $v(\mu)$ with respect to μ and evaluated at ν and $[P,P,\ldots,P] \in \mathbb{C}^{q\times q}$, see also [14,18].

Let $\mathscr{T}$ be the set of all formal power series $T(x,\mu)$ with no constant terms, i.e., $\mathbb{F}[\![x,\mu]\!] = \mathbb{F} \oplus \mathscr{T}$. The near-identity time rescaling map is given by

$$t = \theta_T(\tau) = \tau + T(x,\mu)\tau, \quad \text{where } T(x,\mu) \in \mathscr{T}. \tag{3.5}$$

The operator Θ_T generated by T sends the vector field v into a vector field given by

$$(\Theta_T v)(x,\mu) = \frac{d\theta_T}{d\tau} = \big(1 + T(x,\mu)\big)v(x,\mu).$$

In order to consider linear change of variables, let $GL(n)$ denote $n \times n$ invertible matrices. Thus, $GL(p)$ and $GL(q)$ are generators of invertible linear change of state variables and invertible linear reparametrization, respectively. One can consider the cases with a countable infinite number of parameters to treat formal versal deformations of a system with infinite number of versal parameters; this is out of the scope of this paper. Therefore, for any $B \in GL(p)$ and $C \in GL(q)$ we have

$$x = \gamma_B(y) = By, \qquad \Gamma_B(v)(y,\mu) = B^{-1}v(By,\mu), \tag{3.6}$$

and

$$\mu = \lambda_C \nu = C\nu \quad \text{and} \quad (\Lambda_C v)(x,\nu) = v(x,C\nu). \tag{3.7}$$

Linear changes of state variables are usually used to put the linear part of the system into a certain normal form, e.g., Jordan canonical form. In Sections 5–7, we assume that the linear part is already in the desired form, and thus, linear change of state variable will not be used in our normal form computations.

Remark 3.1. All systems and changes of variables here are presented in formal power series, and we treat them without any concern about their convergence, see [26] for a powerful analogous result on analytic vector fields. One can also consider the formal vector fields as germs of smooth vector fields and by using the Borel–Ritt Lemma try to extend the results from formal normal forms to smooth normal forms; which however are not discussed in this paper, see, e.g., [12,21].

A sequence of polynomial generators for the state, time and parameter changes, i.e., Eqs. (3.1)–(3.5), are generally used and consecutively applied to normalize a system. Thus, they must be very well coordinated in a systematic approach and their composition is important.

Let $\mathscr{L}$ denote the vector space span of all vector fields having linear part Ax, where A is a fixed non-zero matrix. This implies that $\mathscr{L}$ consists of vector fields v with a linear part cAx for some $c \in \mathbb{F}$. In Section 5 we consider the case $A = 0$ and $\mathscr{L}$ will be defined differently. The spaces of near-identity change of variables and their associated operators make up a group structure as follows. We use the following notations:

$$\mathcal{S} := \{\Phi_S \mid S \in \mathscr{S}\}, \qquad \mathcal{P} := \{\Psi_P \mid P \in \mathscr{P}\} \quad \text{and} \quad \mathcal{T} := \{\Theta_T \mid T \in \mathscr{T}\}.$$

The group action for the state variable follows $\Phi_{S_1} \circ \Phi_{S_2} = \Phi_{S_1 * S_2}$, where $S_1 * S_2$ follows the Campbell–Baker–Hausdorff formula. For time and parameter, it is a straightforward calculation. All these operators act on the vector fields and are subgroups of the group of formal invertible transformations acting on vector fields in $\mathscr{L}$. So, let

$$\mathscr{G}(p) := \bigl\{\Gamma_B \bigm| B \in GL(p)\bigr\}, \tag{3.8}$$

$$\mathscr{H}(q) := \bigl\{\Lambda_C \bigm| C \in GL(q)\bigr\}, \tag{3.9}$$

$$G_1 := \mathcal{SPT} = \{\Phi_S \circ \Psi_P \circ \Theta_T \mid S \in \mathscr{S},\ P \in \mathscr{P},\ T \in \mathcal{T}\}, \tag{3.10}$$

$$G := \mathscr{G}(p)\mathscr{H}(q)G_1 = \bigl\{\Lambda_C \circ \Gamma_B \circ g \bigm| C \in GL(p),\ B \in GL(q),\ g \in G_1\bigr\}. \tag{3.11}$$

In the following, we prove that G_1 and G are groups and then they are called the group of *near-identity transformations* and *invertible transformations*, respectively. We denote the identity map by 1. Thus, $\Phi_S = 1$ ($\Psi_P = 1$ or $\Theta_T = 1$) if and only if $S = 0$ ($P = 0$ or $T = 0$, respectively).

For any $\Theta_T \in \mathcal{T}, \Phi_S \in \mathcal{S}$ and an arbitrary vector field $v \in \mathscr{L}$ we have

$$\begin{aligned}
\bigl(\Phi_S \circ \Theta_T(v)\bigr)(y, \mu) &= \Phi_S(v)(y, \mu) + \Phi_S(Tv)(y, \mu) \\
&= (D\phi_S)^{-1} v(\phi_S, \mu) + (D\phi_S)^{-1} T(\phi_S, \mu) v(\phi_S, \mu) \\
&= (D\phi_S)^{-1} v(\phi_S, \mu) + T(\phi_S, \mu)(D\phi_S)^{-1} v(\phi_S, \mu) \\
&= \Theta_{T(\phi_S, \mu)}\bigl((D\phi_S)^{-1} v(\phi_S, \mu)\bigr) = \bigl(\Theta_{\tilde{T}} \circ \Phi_S(v)\bigr)(y, \mu),
\end{aligned} \tag{3.12}$$

where $\tilde{T}(y, \mu) := T(\phi_S(y, \mu), \mu)$. Thus, $\mathcal{T}$ is a subgroup of $\mathcal{TS}$. By

$$\begin{aligned}
\bigl(\Psi_P \circ \Phi_S(v)\bigr)(y, \nu) &= \Psi_P (D\phi_{S(y,\nu)})^{-1} v(\phi_{S(y,\nu)}, \nu) \\
&= (D\phi_{S(y,\psi_P)})^{-1} v(\phi_{S(y,\psi_P)}, \psi_P) \\
&= \Phi_{S(y,\psi_P)}(v)(y, \psi_P) = \bigl(\Phi_{\hat{S}} \circ \Psi_P(v)\bigr)(y, \mu),
\end{aligned} \tag{3.13}$$

for $\hat{S}(y, \mu) := S(y, \psi_P(\mu))$, and by denoting $\hat{T}(x, \nu) := T(x, \psi_P(\nu))$ we have

$$\begin{aligned}
\bigl(\Psi_P \circ \Theta_T(v)\bigr)(x, \nu) &= \Psi_P(v)(x, \nu) + \Psi_P(Tv)(x, \nu) \\
&= v(x, \psi_P) + T(x, \psi_P) v(x, \psi_P) \\
&= \Theta_{T(x,\psi_P)}(v)(x, \psi_P) = \bigl(\Theta_{\hat{T}} \circ \Psi_P(v)\bigr)(x, \nu).
\end{aligned} \tag{3.14}$$

Now for any $C \in GL(q)$ and $B \in GL(p)$, and by denoting $\overline{S}(y, \mu) := S(y, \lambda_C \nu)$ and $\overline{T}(x, \nu) := T(x, \lambda_C \nu)$ we obtain

$$\begin{aligned}\Lambda_C \circ \Phi_S(v)(y,\nu) &= (D\phi_{S(y,\lambda_C\nu)})^{-1} v(\phi_{S(y,\lambda_C\nu)}, \lambda_C\nu)\\ &= \Phi_{S(y,\lambda_C\nu)}(v)(y,\lambda_c\nu) = \big(\Phi_{\bar{S}} \circ \Lambda_C(v)\big)(y,\nu),\end{aligned} \tag{3.15}$$

$$\begin{aligned}\Lambda_C \circ \Theta_T(v)(x,\nu) &= \Lambda_C(v)(x,\nu) + \Lambda_C(Tv)(x,\nu)\\ &= v(x,\lambda_C\nu) + T(x,\lambda_C\nu)v(x,\lambda_C\nu)\\ &= \Theta_{T(x,\lambda_C\nu)}(v)(x,\lambda_C\nu) = \big(\Theta_{\bar{T}} \circ \Lambda_C(v)\big)(x,\nu),\end{aligned} \tag{3.16}$$

$$\begin{aligned}\Lambda_C \circ \Psi_P(v)(x,\nu) &= \Lambda_C(v)\big(x,\psi_P(\nu)\big)\\ &= v\big(x,\psi_P(C\nu)\big) = \big(\Psi_{\lambda_C(C^{-1}P)} \circ \Lambda_C(v)\big)(x,\nu).\end{aligned} \tag{3.17}$$

Further, let $\breve{S}(y,\mu) := S(By,\mu)$ and $\breve{T}(y,\mu) := T(\gamma_B y,\mu)$. Then, we have $(D\phi_{\breve{S}})(y,\mu) = (D\phi_S)_{(By,\mu)}B = (D\phi_{SB})_{(By,\mu)}$ and

$$\begin{aligned}\Gamma_B \circ \Phi_S(v)(y,\mu) &= B^{-1}\big((D\phi_S)_{(By,\mu)}\big)^{-1} v(\phi_{S(By,\mu)},\mu)\\ &= (D\phi_{\breve{S}(y,\mu)})^{-1} v(\phi_{\breve{S}(y,\mu)},\mu) = \Phi_{\breve{S}}(v)(y,\mu),\end{aligned} \tag{3.18}$$

$$\begin{aligned}\Phi_S \circ \Gamma_B(v)(y,\mu) &= (D\phi_S)^{-1} B^{-1} v(B\phi_{S(y,\mu)},\mu)\\ &= (D\phi_{BS(y,\mu)})^{-1} v(\phi_{BS(y,\mu)},\mu) = \Phi_{BS}(v)(y,\mu),\end{aligned} \tag{3.19}$$

$$\begin{aligned}\Gamma_B \circ \Theta_T(v)(y,\mu) &= \Gamma_B(v)(y,\mu) + \Gamma_B(Tv)(y,\mu)\\ &= B^{-1}v(\gamma_B y,\mu) + B^{-1}T(\gamma_B y,\mu)v(\gamma_B y,\mu)\\ &= \Theta_{B^{-1}T(\gamma_B y,\mu)B}\big(B^{-1}v\big)(\gamma_B y,\mu) = \big(\Theta_{B^{-1}\breve{T}B} \circ \Gamma_B(v)\big)(x,\nu),\end{aligned} \tag{3.20}$$

$$\begin{aligned}\Gamma_B \circ \Psi_P(v)(y,\nu) &= \Gamma_B(v)\big(y,\psi_P(\nu)\big)\\ &= B^{-1}v\big(\gamma_B y,\psi_P(\nu)\big) = \big(\Psi_P \circ \Gamma_B(v)\big)(y,\nu).\end{aligned} \tag{3.21}$$

Note that Eqs. (3.12)–(3.21) imply that G is a group and G_1 is its subgroup.

Remark 3.2. Any $g \in G$ acts linearly on $\mathscr{L}$, i.e., for any $a \in \mathbb{R}$ and formal vector fields v and w we have $g(av + w) = agv + gw$. Therefore, G is a subgroup of the vector space automorphisms of $\mathscr{L}$. Further, Assumption 6.3 given in [9] also holds.

Experience shows that it is useful to consider various gradings in different problems, e.g., see [2, 7,14,15,17]. For each nonnegative integer vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_q) \in \mathbb{N}_0^q$ we define three gradings, all denoted by δ_α, on the three spaces $\mathscr{S}$, $\mathscr{P}$ and $\mathscr{T}$. For any monomial vector $x^{\mathbf{n}}\mu^{\mathbf{m}}\mathbf{e}_i \in \mathscr{S}$ (for $\mathbf{e}_i \in \mathbb{F}^p$), we define $\delta_\alpha(x^{\mathbf{n}}\mu^{\mathbf{m}}\mathbf{e}_i) = |\mathbf{n}| + \alpha \cdot \mathbf{m} - 1$, where $\alpha \cdot \mathbf{m} = \sum_{i=1}^q \alpha_i m_i$. When $\mu^{\mathbf{m}}\mathbf{f}_i \in \mathscr{P}$ (for $\mathbf{f}_i \in \mathbb{F}^q$), $\delta_\alpha(\mu^{\mathbf{m}}\mathbf{f}_i) = \alpha \cdot \mathbf{m} - \alpha_i$, and finally for $x^{\mathbf{n}}\mu^{\mathbf{m}} \in \mathscr{T}$, $\delta_\alpha(x^{\mathbf{n}}\mu^{\mathbf{m}}) = |\mathbf{n}| + \alpha \cdot \mathbf{m}$. For each problem, a choice of α will be stated but it will be the same for all of the three gradings. The spectral sequence construction in Section 4 depends on a choice of grading and automatically gives the various kinds of normal forms obtained by Baider and Sanders [7], Algaba et al. [2], and Kokubu et al. [17] as soon as the values of α_i's are specified. For simplicity an integer subscript, say k, is used to denote the subspace of all homogeneous terms of grade k, e.g., $\mathscr{L}_k$, $\mathscr{T}_k$, $\mathscr{P}_k$, etc. (This should not be confused with E_r in Section 4, in which r stands for the level of E-terms.) This results in graded structures and their associated filtration needed throughout this paper. We have $[\mathscr{L}_k, \mathscr{L}_l] \subseteq \mathscr{L}_{k+l}$, $\mathscr{T}_k\mathscr{T}_l \subseteq \mathscr{T}_{k+l}$ and $\mathscr{T}_k\mathscr{L}_l \subseteq \mathscr{L}_{k+l}$ for any $k, l \in \mathbb{N}_0$.

Remark 3.3. We can think of $\mathbf{e}_i = \frac{\partial}{\partial x_i}$ and $\mathbf{f}_j = \frac{\partial}{\partial \mu_j}$. Since x_i has degree 1, $\frac{\partial}{\partial x_i}$ has degree -1. Since μ_j has degree α_j, $\frac{\partial}{\partial \mu_j}$ has degree $-\alpha_j$. With this convention, all three gradings δ_α follow the same form.

Note that G_1 is not merely a subgroup of the automorphisms of $\mathscr{L}$ but also a subgroup of the filtration-preserving automorphisms, i.e., $\Phi_S(\mathscr{F}^n\mathscr{L}) \subseteq \mathscr{F}^n\mathscr{L}$ for $n \in \mathbb{N}_0$ and $S \in \mathscr{S}$. (This is similar for reparametrization and time rescaling.) Given the gradings and their associated filtration, we now present G as a filtration-preserving group in terms of its subgroups. The following also generalizes the results of Baider and Churchill [6, Section 3] that in terms of our notations it indicates $\mathcal{S}\mathscr{G}(p) = \mathscr{G}(p) \rtimes_\gamma \mathcal{S}$ ($\rtimes$ denotes semidirect product), where no parameter was involved and γ was restricted to $\mathcal{S}$.

Theorem 3.4. *There exist group homomorphisms σ, β and ρ such that $G_1 = (\mathcal{T} \rtimes_\sigma \mathcal{S}) \rtimes_\beta \mathcal{P}$ and $G = (\mathscr{H}(q) \times \mathscr{G}(p)) \rtimes_\rho G_1$. Further, G is a subgroup of filtration-preserving automorphisms of $\mathscr{L}$.*

Proof. Define

$$\sigma : \mathcal{S} \to \operatorname{Aut}(\mathcal{T}),$$
$$\Phi_S \mapsto \sigma(\Phi_S),$$

by $\sigma(\Phi_S)\Theta_T := \Phi_S \circ \Theta_T \circ \Phi_S^{-1} = \Theta_{\tilde{T}}$ for any $T \in \mathscr{T}$, where $\tilde{T}(y, \mu) = T(\phi_S(y, \mu), \mu)$. Campbell–Baker–Hausdorff formula proves that σ is a group homomorphism. Eq. (3.12) implies that $\mathcal{TS} = \mathcal{ST}$ and $\mathcal{TS}$ is a group. Further, we have $\mathcal{T} \cap \mathcal{S} = \{1\}$. These prove $\mathcal{TS} = \mathcal{T} \rtimes_\sigma \mathcal{S}$. It is easy to see that $\mathcal{P} \cap \mathcal{S} = \mathcal{T} \cap \mathcal{P} = \{1\}$. By Eqs. (3.13)–(3.14) we have $\mathcal{S} \lhd \mathcal{PS}$ (i.e., $\mathcal{S}$ is a normal subgroup of $\mathcal{PS}$), $\mathcal{T} \lhd G_1$, and $\mathcal{ST} \lhd G_1$.

Now we claim $\mathcal{P} \cap \mathcal{ST} = \{1\}$. By contradiction we assume $1 \neq g \in \mathcal{P} \cap \mathcal{ST}$. Then, there exist non-zero elements $S \in \mathscr{S}$, $P \in \mathscr{P}$ and $T \in \mathscr{T}$ such that $g = \Psi_P = \Theta_T \Phi_S$. Thereby, there exists an i ($0 < i \leqslant q$) such that the i-th component of P, say P_i, is non-zero. So, for any natural number k we have $\Psi_P(\mu_i^k S(x, 0)) = (\mu_i + P_i)^k S(x, 0)$ while

$$\Theta_T \circ \Phi_S\big(\mu_i^k S(x, 0)\big) = \Theta_T\big(\mu_i^k S(x, 0)\big) = \big(\mu_i^k + \mu_i^k T\big) S(x, 0).$$

This results in $(\mu_i + P_i)^k = \mu_i^k + \mu_i^k T$. Recall that P and T have no constant or linear terms. So, the highest degree of both sides are $(P_i)^k$ and $\mu_i^k T$. Hence, $(P_i)^k = \mu_i^k T$ for any $k \in \mathbb{N}$. This is a contradiction and therefore, $\mathcal{P} \cap \mathcal{ST} = \{1\}$. Thus, the formulas (3.13) and (3.14) imply $\mathcal{ST} \lhd \mathcal{P}(\mathcal{ST})$. Next, define the map

$$\beta : \mathcal{P} \to \operatorname{Aut}(\mathcal{TS}),$$
$$\Psi_P \mapsto \beta(\Psi_P),$$

by $\beta(\Psi_P)\Theta_T := \Psi_P \circ \Theta_T \circ \Psi_P^{-1} = \Theta_{\hat{T}}$ and $\beta(\Psi_P)\Phi_S := \Psi_P \circ \Phi_S \circ \Psi_P^{-1} = \Phi_{\hat{S}}$, where $\hat{T}(x, \nu) = T(x, \psi_P(\nu))$ and $\hat{S}(x, \nu) = S(x, \psi_P(\nu))$. Then, β is a group homomorphism and thus, we have $G_1 = \mathcal{TS} \rtimes_\beta \mathcal{P}$. Eqs. (3.15)–(3.21) infer that $\mathscr{H}(q) \times \mathscr{G}(p) = \mathscr{H}(q)\mathscr{G}(p) \lhd G$. Now define

$$\rho : \mathscr{H}(q) \times \mathscr{G}(p) \to \operatorname{Aut}(G_1)$$

by $\rho((\Lambda_C, \Gamma_B)) = \rho(\Lambda_C) \circ \rho(\Gamma_B)$, and

$$\rho(\Lambda_C)(\Psi_P \circ \Theta_T \circ \Phi_S) := \Lambda_C \circ \Psi_P \circ \Theta_T \circ \Phi_S \circ \Lambda_C^{-1} = \Psi_{CP} \circ \Theta_{\check{T}} \circ \Phi_{\check{S}},$$

where $\breve{T}(x,\nu)=T(x,\lambda_C\nu)$, $\breve{S}(x,\nu)=S(x,\lambda_C\nu)$ and

$$\rho(\Gamma_B)(\Psi_P\circ\Theta_T\circ\Phi_S):=\Gamma_B\circ\Psi_P\circ\Theta_T\circ\Phi_S\circ\Gamma_B^{-1}=\Psi_P\circ\Theta_{\overline{T}}\circ\Phi_{\overline{S}},$$

with $\overline{T}(y,\mu)=T(\gamma_B y,\mu)$ and $\overline{S}(y,\mu)=S(\gamma_B y,\mu)$. It is easy to verify that ρ is a group homomorphism and thus, $G=(\mathscr{H}(q)\times\mathscr{G}(p))\rtimes_\rho G_1$. Furthermore, each generator of the group G preserves filtration.

This completes the proof. $\square$

Theorem 3.4 explains how any finite number of transformations can be composed together, i.e., for any $g\in G$, there exist B, C, T, S, and P such that $g=\Lambda_C\Gamma_B\Theta_T\Phi_S\Psi_P$. This theorem also implies that none of the three near-identity transformations can be obtained from the other two. This is consistent with our claim that all of the three near-identity transformations are needed for parametric normal forms. It also explains why normal forms alongside with their orbital normal forms have been considered in the literature.

Let

$$\mathscr{F}^m\mathcal{S}=\{\Phi_S\mid S\in\mathscr{F}^m\mathscr{S}\},\qquad \mathscr{F}^m\mathcal{T}=\{\Theta_T\mid T\in\mathscr{F}^m\mathscr{T}\}\quad\text{and}\quad \mathscr{F}^m\mathcal{P}=\{\Psi_P\mid P\in\mathscr{F}^m\mathscr{P}\}$$

for any $m\in\mathbb{N}$. Thus, the following holds.

Corollary 3.5. *For any $v\in\mathscr{F}^k\mathscr{L}$ and $g\in\mathscr{F}^mX$ (X denotes either of $\mathcal{P}$, $\mathcal{T}$ and $\mathcal{S}$) define $(g-1)(v)=gv-v$. Then, we have*

$$\mathscr{F}^lX=\mathscr{F}^pX\mathscr{F}^qX,\quad \text{where } l=\min\{p,q\},\tag{3.22}$$

$$\pi_{\mathscr{L}_s}(gv)=\pi_{\mathscr{L}_s}(v)\quad \text{for } s<k+m,\tag{3.23}$$

$$(\mathscr{F}^mX-1)(\mathscr{F}^k\mathscr{L})\subseteq\mathscr{F}^{m+k}\mathscr{L}.\tag{3.24}$$

Now we follow [9] to define initially linear group actions. Let V and W be two graded vector spaces, $h:V\to W$ and $h=h_L+h_H$. We assume that h_L and h_H are filtration-preserving maps, i.e., $h_L(\mathscr{F}^pV)\subseteq F^pW$ and $h_H(\mathscr{F}^pV)\subseteq F^pW$. Here, L stands for linear and H stands for higher order. Further, assume h_L is linear and for any $v\in V$,

$$h_Lv\in\mathscr{F}^pW\quad\Longrightarrow\quad h_Hv\in\mathscr{F}^{p+1}W.\tag{3.25}$$

Then, we say that h is an initially linear map and refer to h_L as its linear part. Now let A^0 be a group, A^1 a vector space and A^0 act on A^1 as a group action on a vector space, i.e., $g:A^1\to A^1$ is linear for any $g\in A^0$. This group action is called *initially linear* when the map $h^g:A^1\to A^1$ (given by $h^g(v)=gv-v$ for any $v\in\mathscr{L}$) is an initially linear map.

Remark 3.6. Note that any $g\in\mathcal{P}$ does not necessarily act as an initially linear map on $\mathscr{L}$, see Remark 7.2. For example, consider $p=1$, $q=2$, $v=x(\mu_1^2+\mu_2^2)$ and $P=[-\mu_2^2,\mu_1\mu_2]$. Then,

$$D_\mu\big(v(\mu),P\big)=\mathbf{0}\quad\text{and}\quad \Psi_Pv(\mu)=\frac{1}{2}D_\mu^2\big(v(\mu),P\big)=x\mu_2^2\big(\mu_1^2+\mu_2^2\big)\neq\mathbf{0}.$$

This implies that Ψ_P is not initially linear. This is because its linear part, $D_\mu(v(\mu),\cdot)$, is neither an *injective* nor a bijective linear map. Our method was applied in [13] on a generalized Hopf singularity and the system was called parametric generic system where this map was injective, see also Remark 7.6. We work on a vector field v which is called a non-degenerated deformation for the vector field $v(x,\mathbf{0})$, where this map is onto, see also [11]. Instead of a general definition, we shall formally define non-degenerated deformation for individual vector fields associated with single zero, resonant saddle, and Hopf singularities discussed in Sections 5–7, respectively.

4. Cohomology spectral sequences

In this section, we present a quick review on the spectral sequence method and its relation with parametric normal forms (see also [20,35], and for a more detailed discussion on normal forms, see [9,22,27–29]). The reader can simply assume the formula (4.5) as a systematic method for tracking normal forms and their uniqueness through spectral sequences method and skip the first two pages of this section.

Spectral sequences (of cohomological type) in general are a page-sequence of left R-modules (where R is a commutative ring with identity) which comes with a page-sequence of differentials and each page is the cohomology of the previous page as follows. Recall that we express most isomorphisms with equality throughout this paper. Consider the cochain complex

$$\cdots \to A^{n-1} \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} A^{n+1} \to \cdots \tag{4.1}$$

where A^n is an R-module, d^n is R-linear and $d^n d^{n-1} = 0$ for $n \in \mathbb{Z}$. This is equivalent to considering $A^* = \bigoplus_n A^n$,

$$d^* : A^* \to A^* \quad \text{with } d^n = d^*|_{A^n} : A^n \to A^{n+1}$$

(i.e., d^* has a degree of $+1$) and $d^* \circ d^* = 0$. (A^*, d^*) is called a *differential graded R-module.*

Denote by $H^n(A^*, d)$ the cohomology of d^* at grade n, i.e., $H^n(A^*, d^*) := \ker d^n / \operatorname{Im} d^{n-1}$. Now assume that A^* comes equipped with a filtration $\mathscr{F}$ compatible with the differential d^*; that is,

$$\cdots \subseteq \mathscr{F}^{n+1} A^* \subseteq \mathscr{F}^n A^* \subseteq \cdots \subseteq A^*,$$

and $d^* : \mathscr{F}^n A^* \to \mathscr{F}^n A^*$ for $n \in \mathbb{Z}$. Then, $(A^*, d, \mathscr{F})$ is called a *filtered differential graded R-module.* For simplicity, assume that filtration comes from a grading structure, that is, $A^{*,*}$ is a bi-graded R-module

$$A = A^{*,*} = \bigoplus_m A^{*,m} = \bigoplus_n A^{n,*}, \quad A^{*,m} = \bigoplus_n A^{n,m},\ A^{n,*} = \bigoplus_m A^{n,m},$$

and $\mathscr{F}^m A^{*,*} = \bigoplus_{k \geqslant m} A^{*,k}$. Thereby, the filtration is Hausdorff and exhaustive.

Spectral sequences are aimed at computing $H^*(A, d) = \bigoplus_m H^m(A, d)$, where H^m denotes the cohomology of the differential d at degree m. In the context of normal form theory $H^*(A, d)$ can be interpreted as the space of unique normal forms, when the graded differential (A, d) is properly chosen. This is clearly demonstrated by Sanders [27,28]. Although

$$H^*(A, d) = \bigoplus_n \bigoplus_m \frac{\mathscr{F}^n H^m(A, d)}{\mathscr{F}^{n+1} H^m(A, d)} = \bigoplus_m \bigoplus_n \frac{\mathscr{F}^n H^m(A, d)}{\mathscr{F}^{n+1} H^m(A, d)},$$

it is, in principal, not possible to compute $H^*(A, d)$ directly from this equation. This is why the spectral sequences are designed to compute $E_\infty^{m,n}$ in a systematic approach. Obviously, it is important to properly define the differential, grading structure and the filtration in such a way that $E_r^{*,*}$ converges to $E_\infty^{*,*} = H^*(A, d)$. To describe the spectral sequences, we denote

$$E_0^{m,n} := \frac{\mathscr{F}^m A^{m+n}}{\mathscr{F}^{m+1} A^{m+n}} = A^{m+n,m}$$

and define the differential d_0 with degree $+1$ by $d_0^{m,*} = d|_{E_0^{m,*}} : E_0^{m,*} \to E_0^{m,*}$. Since $E_r^{m,*} = \bigoplus_n E_r^{m,n}$ is isomorphic to a quotient space of $A^{*,m}$, we can inductively define the r-th level differential of bi-degree $(r, 1-r)$ by

$$d_r^{m,*} = d|_{E_r^{m,*}}, d_r^{m,n} : E_r^{m,n} \to E_r^{m+r,n+1-r}, \quad \text{and also} \quad E_{r+1}^{m,n} = H^n\big(E_r^{m,*}, d_r^{m,*}\big).$$

The filtration $\mathscr{F}$ is called strongly convergent in the sense of Cartan and Eilenberg [20, p. 69] if the filtration is Hausdorff and exhaustive and we have

$$H^*(A,d) = \lim_{\leftarrow n} \frac{H^*(A,d)}{\mathscr{F}^n H^*(A,d)}, \tag{4.2}$$

where $\lim_{\leftarrow n}$ stands for the projective (or inverse) limit. Since

$$\lim_{\leftarrow n} \frac{H^*(A,d)}{\mathscr{F}^n H^*(A,d)} = \bigoplus_{n=0}^{\infty} \frac{\mathscr{F}^n H^*(A,d)}{\mathscr{F}^{n+1} H^*(A,d)},$$

our filtration automatically satisfies the condition (4.2). Therefore, the associated spectral sequence $E_r^{j,k}$ is given by

$$E_r^{m,n} = \frac{Z_r^{m,n}}{Z_{r-1}^{m+1,n-1} + B_{r-1}^{m,n}} \quad \text{and} \quad E_\infty^{m,n} := \frac{Z_\infty^{m,n}}{Z_\infty^{m+1,n-1} + B_\infty^{m,n}}, \tag{4.3}$$

where

$$Z_r^{m,n} := \mathscr{F}^m A^{m+n,*} \cap d^{-1} \mathscr{F}^{m+r} A^{m+n+1,*} \quad \text{and} \quad B_r^{m,n} := \mathscr{F}^m A^{m+n,*} \cap d\big(\mathscr{F}^{m-r} A^{m+n-1,*}\big).$$

Therefore, $Z_\infty^{m,n} := \ker d \cap \mathscr{F}^m A^{m+n,*}$ and $B_\infty^{m,n} := \operatorname{Im} d \cap \mathscr{F}^m A^{m+n,*}$. Furthermore, $E_r^{m,n}$ strongly converges to $H^*(A,d)$, that is,

$$E_\infty^{j,k} = \frac{\mathscr{F}^j H^{j+k}(A,d)}{\mathscr{F}^{j+1} H^{j+k}(A,d)},$$

according to [20, Theorems 2.6, 3.2, 3.12] and that the filtered module $(A, \mathscr{F})$ is Hausdorff and exhaustive. Finally, we call a spectral sequence *collapses at level* r, when $E_n^{*,*} = E_r^{*,*}$ for any $n \geqslant r$; this means that in order to obtain $E_\infty^{*,*}$ we only need to compute $E_r^{*,*}$.

In this paper, the cochain complex (4.1) is just a short cochain complex given by

$$0 \hookrightarrow A^0 \xrightarrow{d} A^1 \to 0, \tag{4.4}$$

where A^0 represents the space of change of variables and A^1 represents the space in which the vector fields live. This short cochain complex simplifies the formulas for $E_r^{*,*}$-terms in Eq. (4.3) as follows:

$$E_r^{m,n} = \begin{cases} 0, & \text{where } n \neq -m, -m+1, \\ (\mathscr{F}^m A^0 \cap d^{-1} \mathscr{F}^{m+r} A^1)/(\mathscr{F}^{m+1} A^0 \cap d^{-1} \mathscr{F}^{m+r} A^1), & \text{where } n = -m, \\ \mathscr{F}^m A^1/(d(\mathscr{F}^{m-r+1} A^0) \cap \mathscr{F}^m A^1 + \mathscr{F}^{m+1} A^1) & \text{when } n = -m+1 \end{cases} \tag{4.5}$$

(for an illuminating presentation and the proof see [9]). One may simply assume the formula (4.5) as a systematic method for tracking normal forms and their uniqueness. Indeed, the r-th level normal forms of vector fields live in $\bigoplus_n E_r^{n,-n+1}$ while their available symmetries (in the transformation space) live in $\bigoplus_n E_r^{n,-n}$.

Now we demonstrate the role of spectral sequences method in computation of normal form theory. Recalling the identification in Lemma 2.1, one can correlate normal form computations and the consecutive quotients appearing in the spectral sequence method. Indeed, it indicates the terms which may or may not be eliminated from the system, i.e., the $\mathscr{B}$-space U represents the terms that are being eliminated while the $\mathscr{B}$-complement space identified with the quotient space represents terms which may not be eliminated. Note that this is not to say that terms in the $\mathscr{B}$-complement space will not be changed (or accidentally eliminated) at this step. The other motivation for this is the following lemma. So, consider that we are at some step of normal form computation. The idea is to identify the consecutive quotient spaces with descending $\mathscr{B}$-complement spaces. We also wish to identify the updated maps (differentials) simply with the original map being sequentially restricted to the descending $\mathscr{B}$-complement spaces. This is why we need to look at quotient spaces at some point as space of cosets (in homology theory) and at the rest as $\mathscr{B}$-complement spaces (in normal form theory). Let us describe $\mathscr{B}$-spaces $\mathscr{T}_0$, $\mathscr{T}_1$ and $\mathscr{N}_1$ used in the following lemma by their terms as follows. Denote $\mathscr{T}_0$ (that is in the denominator) for terms that had already been eliminated, $\mathscr{N}_1$ for terms not being eliminated at this step (although they can be still modified or accidentally eliminated), and $\mathscr{T}_1$ for terms being eliminated at this step (that will appear in the dominator of the next step). Further, let d and d^* stand for the differential (map) of the previous level and the updated differential, respectively. Recall that $\overline{E/F}$ denotes the $\mathscr{B}$-complement space of F in E in the following lemma to avoid confusion.

Lemma 4.1. *Let V and W be vector spaces over $\mathbb{F}$, $\mathscr{B}=\{e_n\}_{n=1}^{\infty}$ a formal basis or $\mathscr{B}=\{e_n\}_{n=1}^{\dim V}$ a finite ordered basis for V, and $\mathscr{T}_0$ a $\mathscr{B}$-subspace of V. Furthermore, assume that $d: W\to V$ is a linear map satisfying $\pi_{\mathscr{T}_0}\circ d(W)=\mathbf{0}$. Then, d naturally induces linear maps given by $d^*: W\to V/\mathscr{T}_0$ and $\overline{d^*}: W\to \overline{V/\mathscr{T}_0}$. Hence, there exist unique $\mathscr{B}$-subspaces $\mathscr{N}_1$ and $\mathscr{T}_1\subseteq V$ such that*

(1) $\overline{\operatorname{coker}\overline{d^*}}=\overline{V/\mathscr{T}_1}=\overline{(\mathscr{N}_1+\mathscr{T}_1)/\mathscr{T}_1}=\mathscr{N}_1$, *for* $\mathscr{T}_0\subseteq\mathscr{T}_1$ *and* $\mathscr{N}_1\oplus\mathscr{T}_1=V$;
(2) *for any $e_m\in\mathscr{B}$ there exist a unique vector $\hat{w}\in V$ (where $\hat{w}+\mathscr{T}_0=d^*(w)$ for some $w\in W$ and $\pi_{\mathscr{T}_0}\hat{w}=\mathbf{0}$) and unique scalars $a_{n_1},a_{n_2},\ldots,a_{n_N}$ ($n_N\leqslant m$) satisfying $e_m-(\hat{w}+\sum_{k=1}^{N}a_{n_k}e_{n_k})\in\mathscr{T}_0$, where $\mathscr{B}\cap\mathscr{N}_1=\{e_{n_k}\}$;*
(3) *for any $v\in V$ satisfying $\pi_{\mathscr{T}_0}(v)=0$, there exists a vector $w\in W$ such that $v-d(w)\in\mathscr{N}_1$.*

(*These properties ensure that $\mathscr{N}_1$ fulfills the role of complement spaces for the image of the updated map d^*.*)

Proof. First notice that $\overline{d^*}$ is the same map as d, except that now its codomain is $\overline{V/\mathscr{T}_0}$ (the complement of $\mathscr{T}_0$ in V, not as a space of cosets). In particular, $\operatorname{Im} d=\operatorname{Im}\overline{d^*}$ is a subspace of $\overline{V/\mathscr{T}_0}$. Further, $\overline{\operatorname{coker}\overline{d^*}}=\overline{\overline{V/\mathscr{T}_0}/\operatorname{Im}\overline{d^*}}=\overline{\mathscr{T}_0^c/\operatorname{Im}\overline{d^*}}$. Note that $\overline{\mathscr{T}_0^c/\operatorname{Im}\overline{d^*}}$ and $(\operatorname{Im} d\oplus\mathscr{T}_0)^c$ are both $\mathscr{B}$-space and $\pi_{\mathscr{T}_0}d(W)=\{0\}$. So, the condition of an $e_i\in\mathscr{B}$ not being an element of $\operatorname{span}\{e_j\mid j<i\}+\operatorname{Im} d\oplus\mathscr{T}_0$ is equivalent to e_i being neither in $\operatorname{span}\{e_j\mid j<i,\ e_j\in\mathscr{T}_0^c\}+\operatorname{Im} d$ nor in $\mathscr{T}_0$. Thus, $(\operatorname{Im} d\oplus\mathscr{T}_0)^c=\overline{\mathscr{T}_0^c/\operatorname{Im}\overline{d^*}}$. Let $\mathscr{N}_1=(\operatorname{Im} d\oplus\mathscr{T}_0)^c$. Since $\mathscr{N}_1=\operatorname{coker}\overline{d^*}$, $\mathscr{N}_1$ is a unique $\mathscr{B}$-space. The condition $\mathscr{N}_1\oplus\mathscr{T}_1=V$ requires that $\mathscr{T}_1=\mathscr{N}_1^c=((\operatorname{Im} d\oplus\mathscr{T}_0)^c)^c$ be a unique $\mathscr{B}$-space satisfying condition (1). Therefore, $(\operatorname{Im}\overline{d^*}\oplus\mathscr{T}_0)\oplus\mathscr{N}_1=V$. Moreover, for any $e_m\in\mathscr{B}$, there exist $\overline{d^*}(w)=\hat{w}\in\operatorname{Im}\overline{d^*}$ and $v_0\in\mathscr{T}_0$ such that $e_m-\hat{w}-v_0\in\mathscr{N}_1$. Thus, $\overline{d^*}(w)+\mathscr{T}_0=d^*(w)$ for which $\pi_{\mathscr{T}_0}(\overline{d^*}(w))=0$. So, we have unique scalars $a_{n_1},a_{n_2},\ldots,a_{n_N}$ ($n_N\leqslant m$) satisfying $e_m-\hat{w}-\sum a_{n_k}e_{n_k}=v_0\in\mathscr{T}_0$, where $\{e_{n_k}\}=\mathscr{B}\cap\mathscr{N}_1$. (3) is straightforward by (2). $\square$

Next, we follow Benderesky and Churchill [9] to present spectral sequences as a method of normal form computations. They are associated with a spectral sequence having any orbit $\{gv\mid g\in\mathcal{S}\}$ for $v\in A^1=\mathscr{L}$. Then, $d_v: A^0\to A^1$ for a short cochain complex is defined by $d_v(g)=h_L^g(v)$, where h_L^g denotes the linear part of the map $g=\Phi_S$ for any $S\in A^0$. For the case of parametric normal forms without linear change of state variable and reparametrization, A^0 corresponds to the generators of the group

$$G_1:=(\mathcal{T}\rtimes_\theta\mathcal{S})\rtimes_\beta\mathcal{P}. \tag{4.6}$$

Let $v = Ax + \text{h.o.t.}$ be the vector field in Eq. (1.1). We recall that $\mathscr{L}$ is the vector space span of all formal vector fields having a linear part of the form Ax, when A is fixed. Define

$$A^{1,*} := \mathscr{L}, \qquad A^{0,*} := \mathscr{S} \times \mathscr{P} \times \mathscr{T} \quad \text{and} \quad A^{*,*} = A^{0,*} \oplus A^{1,*},$$

where $A^{j,k} := \{\mathbf{0}\}$ for $j \neq 0, 1$. Note that $A^{0,*}$ is a graded space, since it is the product of graded spaces. Now the linear part of the group action defines the differential $d = d_v$ by

$$0 \to A^{0,*} \xrightarrow{d} \mathscr{L} \to 0,$$
$$d_v(S, P, T) := D_\mu(v)P + Tv + \operatorname{ad}_S v, \tag{4.7}$$

where $\Phi_S \Psi_P \Theta_T \in G_1$. This results in the infinite level parametric normal form of v via computation of E_∞-terms. Then, $(A^{*,*}, d, \mathscr{F})$ is a locally finite graded filtered differential $\mathbb{F}$-vector space. Finally, we define total complexes for E-terms by

$$\operatorname{Total}^0\big(E_r^{*,*}\big) = \bigoplus_m E_r^{m,-m} \quad \text{and} \quad \operatorname{Total}^1\big(E_r^{*,*}\big) = \bigoplus_m E_r^{m,-m+1}. \tag{4.8}$$

The former represents the r-th level conormal form space (space of transformations) and the latter is the space in which the r-th level normal forms live in, see [27,28].

Up to now we have described Benderesky and Churchill's approach on normal forms. The earlier approach of Sanders [27] considered the spectral sequence $E_r^{j,k}$ by allowing $v^{(r)}$ to be updated during the process of normal form, i.e., $E_r^{j,k} = E_r^{j,k}(v^{(r-1)})$. This is the most common and convenient approach in normal form theory. Benderesky and Churchill's results in [9] imply that updating $v^{(r)}$ does not change $E_r^{n,-n+1}$ (by proving that $E_r^{j,k}(v)$ is invariant under the group orbit of v). Therefore, both approaches are equivalent, see Lemma 4.3 and [9, Theorem 6.11]. One should note that the authors in [9] applied the method in the context of matrix normal form theory, that is why they chose not to update the differentials. It, however, is evident that updating the differential substantially reduces the complexity of computation, see, e.g., [13, Example 2.4.4]. Therefore, we use the converging differentials $d_r = d_r^{*,*}(v^{(r)})$ at each new level of the spectral sequence. Thus, the results [20, Theorems 2.6, 3.2, 3.12] are still valid on normal form theory of vector fields as long as we work with initially linear maps. So, our spectral sequence computations use updating differentials.

Now we are ready to present the first level normal form space in terms of the first level spectral sequence.

Lemma 4.2. *For any $v \in \mathscr{L}$, there exists an automorphism associated with $g_1 \in G$ such that it sends v into the first level normal form $\tilde{v}^{(1)} \in \bigoplus_{n=0}^{\infty} \mathscr{N}_n^{(1)} \subset \mathscr{L}$, where*

$$E_1^{-n,n+1} = \frac{\mathscr{L}_n}{\mathscr{T}_n^{(1)}} = \frac{\mathscr{N}_n^{(1)} + \mathscr{T}_n^{(1)}}{\mathscr{T}_n^{(1)}} = \mathscr{N}_n^{(1)} \quad (\textit{for } n \in \mathbb{N}_0)$$

and the formal basis style is used for the complement spaces $\mathscr{N}_n^{(1)}$.

Proof. Following Lemma 4.1, there exist unique complement vector spaces $\mathscr{N}_n^{(1)}$ and $\mathscr{T}_n^{(1)}$ satisfying

$$E_1^{n,-n+1} = \frac{\mathscr{F}^n \mathscr{L}}{\mathscr{F}^n \mathscr{L} \cap d(\mathscr{F}^n A^{0,*}) + \mathscr{F}^{n+1} \mathscr{L}} = \frac{\mathscr{L}_n}{\pi_{\mathscr{L}_n} \circ dA^{0,n}} = \frac{\mathscr{N}_n^{(1)} + \mathscr{T}_n^{(1)}}{\mathscr{T}_n^{(1)}}.$$

Thus, there exists $(S_n, P_n, T_n) \in A^{0,n}$ such that $v_n^{n-1} + \mathscr{T}_n^{(1)} = v_n^n + \mathscr{T}_n^{(1)}$, where $v_n^n \in \mathscr{N}_n^{(1)}$ and $v_n^n = v_n^{n-1} + \pi_{\mathscr{L}_n} dY_n$. Therefore,

$$v^n = \Phi_n\big(v^{n-1}\big) = \Phi_{S_n} \circ \Psi_{P_n} \circ \Theta_{T_n}\big(v^{n-1}\big) = v_0^0 + v_1^1 + \cdots + v_{n-1}^{n-1} + \big(v_n^{n-1} + \pi_{\mathscr{L}_n} dY_n\big) + \cdots,$$

where $\pi_{\mathscr{L}_n} v^n = v_n^n = v_n^{n-1} + D_\mu v_0^{(0)} P_n + T_n v_0^{(0)} + \mathrm{ad}_{S_n} v_0^{(0)}$. By Theorem 3.4, there exists $(S_{(n)}, P_{(n)}, T_{(n)}) \in A^{0,*}$ such that

$$v^n = \Phi_n \cdots \Phi_2 \Phi_1\big(v^0\big) = \Phi_{S_{(n)}} \circ \Psi_{P_{(n)}} \circ \Theta_{T_{(n)}}\big(v^0\big).$$

Corollary 3.5 implies that $(S_{(n)}, P_{(n)}, T_{(n)})$ converges in filtration topology to an element (S, P, T) from $A^{0,*}$ and thus, v^n is also convergent to $\tilde{v}^{(1)} = g_1 v$, where $g_1 := \Phi_S \circ \Psi_P \circ \Theta_T$.

The proof is complete. □

In this paper $\tilde{v}^{(1)}$ (and $\tilde{v}^{(r)}$) is called *the first* (*the r-th*) *level extended partial parametric normal form*, this comes from Murdock's definition in [22, Section 5] where only non-parametric cases were under consideration.

Lemma 4.3. *For any $v \in \mathscr{L}$, there exist parameter solution $P \in \mathscr{P}$, time solution $T \in \mathscr{T}$, and state solution $S \in \mathscr{S}$ such that their associated automorphisms transform v into the r-th level extended partial parametric normal form*

$$\tilde{v}^{(r)} = g_r v \in \bigoplus_{n=0}^{\infty} \mathscr{N}_n^{(r)} \quad \text{where } g_r = \Psi_P \circ \Theta_T \circ \Phi_S,$$

in which $\mathscr{N}_n^{(r)}$ follows Lemma 4.1, i.e., there exists $\mathscr{B}$-space $\mathscr{T}_n^{(r)}$ such that $E_r^{-n,n+1} = \mathscr{L}_n / \mathscr{T}_n^{(r)} = (\mathscr{N}_n^{(r)} + \mathscr{T}_n^{(r)}) / \mathscr{T}_n^{(r)} = \mathscr{N}_n^{(r)}$ for $n \in \mathbb{N}_0$.

Proof. By Lemma 4.2 there exists a unique (since $\tilde{v}^{(1)}$ and $(S, P, T) \in A^0$ follow formal basis style and costyle) vector field $\tilde{v}^{(1)} = g_1 v \in \bigoplus_{n=0}^{\infty} \mathscr{N}_n^{(1)} = \mathscr{N}^{(1)}$, where $\mathrm{Total}^1(E_1^{*,*}) = (\mathscr{N}^{(1)} + \mathscr{T}^{(1)}) / \mathscr{T}^{(1)} = \mathscr{N}^{(1)}$ and $g_1 = \Phi_S \circ \Psi_P \circ \Theta_T$. Now define a new differential $d_1 = d_1^{n,-n}$ induced by $d_{\tilde{v}^{(1)}}(Y_n) = D_\mu(\tilde{v}^{(1)}) P_n + T_n \tilde{v}^{(1)} + \mathrm{ad}_{S_n} \tilde{v}^{(1)}$ (where $Y_n = (S_n, P_n, T_n)$), i.e.,

$$\begin{gathered} \ker d_0^{n,-n} \xrightarrow{d_1^{n,-n}} \operatorname{coker} d_0^{n+1,-n-1}, \\ Y_n + Z_0^{n+1,-n-1} \mapsto d_{\tilde{v}^{(1)}}(Y_n) \quad \big(\mathrm{mod}\ \mathscr{F}^{n+1}\mathscr{L} \cap d\big(\mathscr{F}^{n+1} A^{0,*}\big) + \mathscr{F}^{n+2}\mathscr{L}\big). \end{gathered} \tag{4.9}$$

Note that $\tilde{v}_0^{(1)} = v_0^{(0)}$. Furthermore, $E_2^{n,-n} = Z_2^{n,-n} / Z_1^{n+1,-n-1}$, where

$$Z_2^{n,-n} = \mathscr{F}^n A^{0,*} \cap d_1^{-1} \mathscr{F}^{n+2} A^{1,*} \quad \text{and} \quad Z_1^{n+1,-n-1} = \mathscr{F}^{n+1} A^{0,*} \cap d_1^{-1} \mathscr{F}^{n+2} A^{1,*}.$$

By Lemma 4.2 (similar to proof of Theorem 3.4) we can compose the transformations and then the sequence of their composition is convergent to $g_2 \in G$ with respect to filtration topology. Then, we have $\tilde{v}^{(2)} = g_2 v \in \bigoplus_{n=0}^{\infty} \mathscr{N}_n^{(2)}$, where $\tilde{v}^{(2)}$ is the second level extended partial parametric normal form of v. This confirms our claim for $r = 2$, and therefore the proof is finished by mathematical induction. □

By Lemma 4.3 there exist state solution S_n, parameter solution P_n and time solution T_n such that

$$v^{(n)} := g_n v^{(n-1)} = \sum_{k=k_0}^{\infty} v_k^{(n)}, \quad \text{where } v_k^{(n)} \in \mathscr{N}_k \text{ for } k \leqslant n, \text{ and } g_n = \Phi_{S_n} \circ \Psi_{P_n} \circ \Theta_{T_n} \in G_1.$$

Then, $\{v^{(n)}\}_{n=0}^{\infty} \subset \mathscr{L}$ and $\{g_n\}_n \subset G_1$ are convergent sequences to a vector field $v^{(\infty)} \in \mathscr{L}$ and a $g_\infty \in G_1$ with respect to their filtration topologies. We call $v^{(\infty)}$ an *infinite level parametric normal form*. The above argument leads to the following theorem.

Theorem 4.4. *Let $v = v^{(0)} = \sum_{n=k_0}^{\infty} v_n^{(0)} \in \mathscr{L}$, where $v_n^{(0)} \in \mathscr{L}_n$. Then, there exists a sequence of near-identity maps $\{g_n\}_n \subseteq G_1$ which transforms $v^{(n)}$ to $v^{(n+1)} = g_{n+1} v^{(n)}$ for $n \in \mathbb{N}_0$ in which $v^{(n)}$ converges (with respect to filtration topology) to an infinite level parametric normal form $v^{(\infty)} = \sum_{r=k_0}^{\infty} v_r^{(\infty)}$, where $v_r^{(\infty)} \in \mathscr{N}_r^{(r)}$,*

$$E_\infty^{-r,r+1} = E_r^{-r,r+1} = \frac{\mathscr{L}_r}{\mathscr{T}_r^{(r)}} = \frac{\mathscr{N}_r^{(r)} + \mathscr{T}_r^{(r)}}{\mathscr{T}_r^{(r)}} = \mathscr{N}_r^{(r)}$$

for $r \in \mathbb{N}_0$ and $\mathscr{N}_r^{(r)}$ follows Lemma 4.1. Furthermore, $g_{(n)} = g_n \cdots g_2 g_1$ converges to g_∞ with respect to filtration topology, where g_∞ is associated with $(S, P, T) \in A^{0,*}$, and $v^{(\infty)} = g_\infty v = \Phi_S \circ \Psi_P \circ \Theta_T(v)$.

Proof. For any $p, r \in \mathbb{N}_0$, we have $Z_r^{p,-p+1} = \mathscr{F}^p A^{1,*} = Z_\infty^{p,-p+1}$, while assuming $p - r \leqslant 1$, we have $B_r^{p,-p+1} = \mathscr{F}^p A^{1,*} \cap d(\mathscr{F}^1 A^{0,*}) = B_\infty^{p,-p+1}$. Then,

$$E_r^{r,-r+1} = \frac{Z_r^{r,-r+1}}{Z_{r-1}^{r+1,-r} + B_{r-1}^{r,-r+1}} = E_\infty^{r,-r+1} \quad \text{and} \quad \mathrm{Total}^1\big(E_\infty^{*,*}\big) = \bigoplus_{r=0}^{\infty} E_r^{r,-r+1}.$$

Since $A^{0,*}$ is locally finite, the rest of the proof follows Lemma 4.3 and $v_r^{(\infty)} = \tilde{v}_r^{(r)}$, where $\tilde{v}^{(r)} = \sum_{n=0}^{\infty} \tilde{v}_n^{(r)}$ denotes the r-th level extended partial parametric normal form. □

Remark 4.5. Since linear changes of variables (3.6) and (3.7) are not initially linear, these are not included in our normal form computations through spectral sequences. As noted before, linear changes of state variables are not used in our computations. However, linear reparametrization is essentially necessary for any possible reduction in the number of parameters of a parametric system. We, however, apply this when the vector fields are in a certain level of normal form, say r-th level. This is done without an inclusion of linear reparametrization in the formalism of spectral sequences. Thus, we *redefine* $E_r^{*,*}$-space as the vector space span of all vector fields in the r-th level normal form after a proper linear reparametrization is applied on them. Thus, this violates the main property of spectral sequences; i.e., each level is not a cohomology of the previous level. However, the most important point is that the formulations in the method is still valid for computing the simplest (by using all the available spectral data in the transformation space) parametric normal forms.

5. Parametric single zero singularity

In this section, we apply the method of spectral sequences to a scalar multi-parameter system with single zero singularity. Yu [36] considered this problem with a single parameter and obtained its *simplest* orbital normal form. We consider a system governed by

$$\dot{x} = v(x, \mu) := \sum_{\mathbf{m} \in \mathbb{N}_0^q,\ |\mathbf{m}|>0} a_{1\mathbf{m}} x \mu^{\mathbf{m}} + \sum_{\mathbf{m} \in \mathbb{N}_0^q} a_{2\mathbf{m}} x^2 \mu^{\mathbf{m}} + \sum_{\mathbf{m} \in \mathbb{N}_0^q} a_{3\mathbf{m}} x^3 \mu^{\mathbf{m}} + \cdots, \tag{5.1}$$

with $x \in \mathbb{F}$ (i.e., $p = 1$), $\mu \in \mathbb{F}^q$, (for a fixed $k \in \mathbb{N}$) $a_{k\mathbf{0}} \neq \mathbf{0}$, and $a_{i\mathbf{0}} = 0$ for any $i < k$.

As noted in Section 3, $\mathscr{L}$ will be defined differently in this example, since $A = 0$. Define B as $(k-1) \times q$ matrix whose (i, j) entry is $b_{ij} := a_{i\mathbf{e}_j}$ for $i, j \in \mathbb{N}$, $i < k$ and $j \leqslant q$. We call v a *non-degenerate deformation* of $v(x, \mathbf{0})$ when $\operatorname{rank}(B) = k-1$. Note that we do not consider the case of general non-degenerate deformations that does not necessarily keep the origin as its rest point. Therefore, we can choose an invertible linear reparametrization $\mu = C\nu = C[\nu_1, \nu_2, \ldots, \nu_q]$ such that $(\nu_1, \nu_2, \ldots, \nu_{k-1}, \mathbf{0}) = BC\nu$ and instead consider the system $\Lambda_C v$. Thus, from now on, for convenience we assume that these have already been done, so that the system $v(x, \mu)$ satisfies $a_{i\mathbf{e}_j} = \delta_{ij}$ when $i < k$. In other words we consider the vector fields described by

$$\mathscr{L} := \operatorname{span}\left\{x^k + \sum_{i=1}^{k-1} x^i \mu_i + \text{h.o.t.}\right\}, \tag{5.2}$$

where h.o.t. denotes monomials of degree at least $(k+1)$ in x or degree 2 in μ.

Remark 5.1. The rank condition on B requires $q \geqslant k-1$. Loosely speaking, this means that the system has enough parameters to unfold the unperturbed ($\mu = 0$) system, with the restriction that the origin remains the rest point. The restriction is because Eq. (5.1) does not allow parameters to produce a constant term.

Note that $\mathscr{L}$ constitutes a Lie algebra and is invariant under near-identity formal change of variables. For this example, we put $\alpha = (k, k, \ldots, k)$. Therefore, the gradings δ_α for $\mathcal{S}$, $\mathcal{P}$, $\mathcal{T}$ are respectively given by $\delta_\alpha(x^i \mu^{\mathbf{m}}) := i + k|\mathbf{m}| - 1$, $\delta_\alpha(x^i \mu^{\mathbf{m}}) := i + k|\mathbf{m}|$, and $\delta_\alpha(\mu^{\mathbf{m}} \mathbf{e}_i) := k|\mathbf{m}| - k$ for $\mathbf{e}_i \in \mathbb{F}^q$. The reason that we put the weight of k for the parameters is that any term with parameters would get a grade greater than or equal to k. Hence, homogeneous spaces $\mathscr{L}_i = \{0\}$ for $i \leqslant k-1$ and $\mathscr{L}_{k-1} = \operatorname{span}\{x^k\}$. The set of all monomial vector fields with any fixed order makes a formal basis leading to the following theorem.

Theorem 5.2. *Let $v(x, \mu)$, given by Eq.* (5.1)*, be a non-degenerate deformation of $v(x, \mathbf{0})$. Then, there exists a $k \in \mathbb{N}$ such that the automorphism g_∞ constructed according to Theorem* 4.4 *sends v into the infinite level parametric normal form*

$$\tilde{v}^{(\infty)} = x^k + \sum_{i=1}^{k-1} x^i \mu_i.$$

Proof. Without loss of generality, we assume $v \in \mathscr{L}$. Since $\delta_\alpha(x^k) = k-1$ and $\delta_\alpha(x\mu_i) = k$ for any $i \leqslant q$, $J_{k-2}(v) = 0$, and $E_\infty^{m,-m+1} = 0$ for $m < k-1$. Here, $E_\infty^{k-1,-k+2}$ depicts the terms of grade $k-1$, that is, the least grade term of the system. So, $E_\infty^{k-1,-k+2} = \operatorname{span}\{x^k\}$. Next, $d(\mathscr{F}^{m-r+1} A^0) \cap \mathscr{F}^m \mathscr{L} = \{0\}$ for $r < k$ implies that

$$E_r^{m,-m+1} = \mathscr{L}_m \quad \text{for } r < k \text{ and } k \leqslant m.$$

By $[x^i \mu^{\mathbf{m}}, x^k] = (k-i) x^{i+k-1} \mu^{\mathbf{m}}$, $x^i \mu^{\mathbf{m}}(x^k) = x^{k+i} \mu^{\mathbf{m}}$ (associated with time rescaling), $D_\mu(x^k, P) = 0$ for any $P \in \mathscr{P}$, we have

$$E_\infty^{m,-m+1} = E_k^{m,-m+1} = \operatorname{span}\left\{x^i \mu_i \;\middle|\; m+1 = k+i,\ 1 \leqslant i < k\right\} \quad \text{for } k \leqslant m < 2k-1,$$

and for $j \geqslant k$, $x^j \mu^{\mathbf{m}}$-terms are eliminated by time rescaling at k-th level. Since $D_\mu(x^i \mu_j, P) = x^i P \mathbf{e}_j$ for $P \in \mathscr{P}$, $x^i \mu^{\mathbf{m}}$-terms are eliminated by reparametrization at the $(i+k)$-level normal form when $m \geqslant 2k-1 \geqslant r > k$, and $i \leqslant r-k$, i.e.,

$$E_r^{m,-m+1} = \frac{\mathscr{F}^m A^1}{d(\mathscr{F}^{m-r+1}A^0) \cap \mathscr{F}^m \mathscr{L} + \mathscr{F}^{m+1}\mathscr{L}}$$
$$= \operatorname{span}\{x^i \mu^{\mathbf{m}} \mid m+1 = i + k|\mathbf{m}|,\ r-k < i < k\}.$$

Therefore,

$$E_\infty^{m,-m+1} = E_{2k}^{m,-m+1} = \{0\} \quad \text{for any } m > 2k-2.$$

The proof is completed by Theorem 4.4. □

6. Parametric resonant saddle singularity

In this section, we consider the following system

$$\dot{\mathbf{x}} = v(\mathbf{x}, \mu) := Ax + \text{h.o.t.}, \tag{6.1}$$

where $p = 2$, $A = \left(\begin{smallmatrix} m & 0 \\ 0 & -n \end{smallmatrix}\right)$, $m, n \in \mathbb{N}$, m/n is an irreducible fraction, $v(\mathbf{0}, \mu) = 0$, and h.o.t. stands for higher order terms with respect to $\mathbf{x} = [x, y]$ and $\mu = [\mu_1, \mu_2, \ldots, \mu_q]$. Some related results can be found in [12,19,33]. Denote $X_{ij} := [x^i y^j, 0]$, $Y_{ij} := [0, x^i y^j]$. For this example we put $\alpha = (\beta, \beta, \ldots, \beta)$. Therefore, the gradings δ_α for $\mathcal{S}$, $\mathcal{P}$, $\mathcal{T}$ are respectively given by $\delta_\alpha(X_{ij}\mu^{\mathbf{m}}) = \delta_\alpha(Y_{ij}\mu^{\mathbf{m}}) = i + j + \beta|\mathbf{m}| - 1$, $\delta_\alpha(x^i y^j \mu^{\mathbf{m}}) = i + j + \beta|\mathbf{m}|$ (associated with time rescaling) and $\delta_\alpha(\mu^{\mathbf{m}}\mathbf{f}_i) = \beta|\mathbf{m}| - \beta$ (associated with reparametrization), $\mathbf{f}_i \in \mathbb{F}^q$. $\mathscr{B} = \operatorname{span}\{X_{ij}\mu^{\mathbf{m}}, Y_{ij}\mu^{\mathbf{m}}\}$ forms a formal basis for $\mathscr{L}$, where its ordering must follow rules (I) and (II) and also Y-terms precede X-terms, when they have the same grade.

Lemma 6.1. *For any $\beta \in \mathbb{N}$ and any parametric system given by* (6.1)*, there exists an automorphism $g_1 \in G$ such that it sends v into*

$$\tilde{v}^{(1)} = g_1 v = A\mathbf{x} + \sum_{i \geqslant 0,\ \mathbf{m} \in \mathbb{N}_0^q} b_{i,\mathbf{m}}^{(1)} [0, x^{in} y^{im+1} \mu^{\mathbf{m}}], \tag{6.2}$$

where $b_{0,\mathbf{0}}^{(1)} = 0$.

Proof. Since $x^{in} y^{im} A\mathbf{x} = mX_{in+1,im} - nY_{in,im+1}$ (associated with time rescaling) and

$$[X_{ij}, A\mathbf{x}] = \big(m(1-i) + nj\big)X_{ij},$$
$$[Y_{ij}, A\mathbf{x}] = \big(-mi + n(j-1)\big)Y_{ij}$$

(associated with state transformation) we have

$$d\big(\mathscr{F}^l A^0\big) \cap \mathscr{F}^l \mathscr{L} = \big(\operatorname{span}\big(\big\{X_{ij}\mu^{\mathbf{m}} \bigm| \delta_\beta\big(X_{ij}\mu^{\mathbf{m}}\big) = l,\ (i-1)m \neq jn\big\} \cup \big\{c \bigm| \delta_\alpha(c) = l\big\}$$
$$\cup \big\{Y_{ij}\mu^{\mathbf{m}} \bigm| \delta_\alpha\big(Y_{ij}\mu^{\mathbf{m}}\big) = l,\ (i, j) \neq (in, im+1)\big\}\big) \bmod \big(\mathscr{F}^{l+1}\big)\big),$$

where $c = (mX_{in+1,im} - nY_{in,im+1})\mu^{\mathbf{m}}$. Therefore,

$$E_1^{l,-l+1} = \frac{\mathscr{F}^l A^1}{d(\mathscr{F}^l A^0) \cap \mathscr{F}^l \mathscr{L} + \mathscr{F}^{l+1}\mathscr{L}} = \operatorname{span}\big\{Y_{in,im+1}\mu^{\mathbf{m}} \bigm| \delta_\alpha\big(Y_{in,im+1}\mu^{\mathbf{m}}\big) = l,\ i \in \mathbb{N}\big\}.$$

Then, using Lemma 4.2 completes the proof. □

The following lemma makes a particular choice for β. Then, it follows that the first $(\beta-1)$ levels of spectral sequence remain the same, so do the normal form vector fields. Normal form vector fields are indeed simplified again at their β-th level.

Lemma 6.2. *Let $\tilde{v}^{(1)}$ be the vector field obtained in Lemma* 6.1*, $k=\min\{i \mid b^{(1)}_{i,\mathbf{0}} \neq 0\}$ and $\beta = k(m+n)+1$. Then, $\tilde{v}^{(1)} = \tilde{v}^{(\beta-1)}$, and $\tilde{v}^{(1)}$ can be transformed into the β-th level extended partial parametric normal form*

$$\tilde{v}^{(\beta)} = A\mathbf{x} + b^{(1)}_{k,\mathbf{0}}\big[0, x^{kn}y^{km+1}\big] + \sum_{i<k,\ |\mathbf{m}|\geqslant 1} b^{(1)}_{i\mathbf{m}}\big[0, x^{in}y^{im+1}\mu^{\mathbf{m}}\big] + \sum_{\mathbf{m}\in\mathbb{N}_0^q} b^{(\beta)}_{2k,\mathbf{m}}\big[0, x^{2kn}y^{2km+1}\mu^{\mathbf{m}}\big],$$

where $b^{(\beta)}_{k,\mathbf{0}} = b^{(1)}_{k,\mathbf{0}} \neq 0$.

Proof. The condition $\beta = k(m+n)+1$ implies that $\delta_\alpha(Y_{01}\mu_i) = \beta$ and $\delta_\alpha(Y_{kn,km+1}) = \beta - 1$. Thus, $E_r^{*,*} = E_1^{*,*}$ for any $r<\beta$. This implies that $\tilde{v}^{(1)} = \tilde{v}^{(\beta-1)}$. Furthermore, we have $E_\infty^{k(m+n),-k(m+n)+1} = \operatorname{span}\{Y_{kn,km+1}\}$, and $E_\infty^{l,-l+1} = E_1^{l,-l+1}$ for any $l<\beta$. Now noticing

$$[Y_{ln,lm+1}, Y_{kn,km+1}] = (km-lm)Y_{(ln+kn)(lm+1+km)},$$
$$[X_{kn+1,km}, Y_{kn,km+1}] = knY_{(2kn)(2km+1)} - kmX_{(2kn+1)(2km)},$$

and $kx^{2kn}y^{2km}Ax = -knY_{2kn,2km+1} + kmX_{2kn+1,2km}$, we have

$$\begin{aligned} d\big(\mathscr{F}^{l-\beta+1}A^0\big) \cap \mathscr{F}^l A^1 = \big(&\operatorname{span}\big(\big\{X_{ij}\mu^{\mathbf{m}} \bigm| \delta_\alpha\big(X_{ij}\mu^{\mathbf{m}}\big) = l,\ (i-1)m \neq jn\big\} \cup \big\{c \bigm| \delta_\alpha(c) = l\big\} \\ &\cup \big\{Y_{in,im+1}\mu^{\mathbf{m}} \bigm| i>k,\ \delta_\alpha\big(Y_{in,im+1}\mu^{\mathbf{m}}\big) = l\big\} \\ &\cup \big\{Y_{ij}\mu^{\mathbf{m}} \bigm| \delta_\alpha\big(Y_{ij}\mu^{\mathbf{m}}\big) = l,\ im \neq (j-1)n\big\} \\ &\cup \big\{Y_{kn,km+1}\mu^{\mathbf{m}} \bigm| \mathbf{m} \neq \mathbf{0},\ l = \beta - 1 + \beta|\mathbf{m}|\big\}\big) \bmod \big(\mathscr{F}^{l+1}A^1\big)\big), \end{aligned}$$

where $c = (mX_{in+1,im} - nY_{in,im+1})\mu^{\mathbf{m}}$. Therefore, for any $l \geqslant \beta$ the following holds

$$E_\beta^{l,-l+1} = \operatorname{span}\big(\big\{Y_{in,im+1}\mu^{\mathbf{m}} \bigm| l = \delta_\alpha\big(Y_{in,im+1}\mu^{\mathbf{m}}\big) \text{ and } (i<k \text{ or } i = 2k)\big\}\big).$$

It is easy to see that $b^{(\beta)}_{i\mathbf{m}} = b^{(1)}_{i\mathbf{m}}$ for $i \leqslant k$. The proof is complete. □

Now we introduce the notion of non-degenerate deformations for the obtained β-th level extended partial parametric normal forms. Then, we need linear reparametrization in order to simplify the system and possibly reduce the number of parameters. However, we do not apply the linear reparametrization on the space $E_\beta^{*,*}$ but we rather apply linear reparametrization on the vector fields. Then, we *redefine* the space $E_\beta^{*,*}$ in order to proceed the computation to higher level spaces.

Let the hypotheses in Lemma 6.1 hold. In addition, define

$$B^{(\beta)} := \begin{pmatrix} b^{(\beta)}_{0,\mu_1} & b^{(\beta)}_{0,\mu_2} & \cdots & b^{(\beta)}_{0,\mu_q} \\ b^{(\beta)}_{1,\mu_1} & b^{(\beta)}_{1,\mu_2} & \cdots & b^{(\beta)}_{1,\mu_q} \\ \cdots & \cdots & \cdots & \cdots \\ b^{(\beta)}_{k,\mu_1} & b^{(\beta)}_{k,\mu_2} & \cdots & b^{(\beta)}_{k,\mu_q} \\ b^{(\beta)}_{2k,\mu_1} & b^{(\beta)}_{2k,\mu_2} & \cdots & b^{(\beta)}_{2k,\mu_q} \end{pmatrix}. \tag{6.3}$$

We say that $v(\mathbf{x}, \mu)$ is a *non-degenerate deformation* for $v(\mathbf{x}, \mathbf{0})$ if $\operatorname{rank}(B^{(\beta)}) = k+2$. A linear reparametrization can put $B^{(\beta)}$ into $[I\ \mathbf{0}]$ where I denotes the $(k+2)\times(k+2)$-identity matrix and $\mathbf{0}$ denotes the $(k+2)\times(q-k-2)$-zero matrix. This linear reparametrization sends $\tilde{v}^{(\beta)}$ into

$$\hat{v}^{(\beta)} = A\mathbf{x} + b^{(1)}_{k,\mathbf{0}}\big[0, x^{kn}y^{km+1}\big] + \sum_{i=0}^{k-1}\big[0, x^{in}y^{im+1}\mu_{i+1}\big] + \sum_{i<k,\ |\mathbf{m}|\geqslant 2} b^{(1)}_{i\mathbf{m}}\big[0, x^{in}y^{im+1}\mu^{\mathbf{m}}\big] + \sum_{\mathbf{m}\in\mathbb{N}_0^q} b^{(\beta)}_{2k,\mathbf{m}}\big[0, x^{2kn}y^{2km+1}\mu^{\mathbf{m}}\big]. \tag{6.4}$$

We now build up our spectral sequence on all such vector fields. That is, $E^{*,*}_\beta$ is *redefined to be the vector space span of all such vector fields.*

Theorem 6.3. *Let v be a non-degenerate deformation of resonant saddle singularity vector field governed by Eq. (6.1) and satisfy the hypothesis given in this section. Then, there exists a natural number k such that v can be transformed by invertible transformation into the infinite level extended partial parametric normal form*

$$\tilde{v}^{(\infty)} = \begin{pmatrix} mx \\ -ny + ax^{kn}y^{km+1} + (b+\mu_{k+1})x^{2kn}y^{2km+1} + \sum_{i=0}^{k-1} x^{in}y^{im+1}\mu_{i+1}\end{pmatrix}, \tag{6.5}$$

where $a, b \in \mathbb{R}$ are constants, $a \neq 0$, and μ_i's are scaled parameters.

Proof. Without loss of generality, we may assume v is transformed into $\hat{v}^{(\beta)}$ (for $\beta = k(m+n)+1$) and is governed by Eq. (6.4). We claim that the 2β-th level extended partial parametric normal form is the same as the infinite level given by Eq. (6.5).

Due to

$$\delta_\alpha(Y_{in,im+1}\mu_i) = in + im + \beta, \qquad D_\mu(Y_{in,im+1}\mu_i, P) = \pi_i(P) = P_i,$$

and $\operatorname{rank}(B^{(\beta)}) = k+2$, for any $\beta < r \leqslant (k-1)(m+n)+\beta$ we have

$$\begin{aligned} E_r^{l,-l+1} = \operatorname{span}\Bigg(&\bigg\{Y_{in,im+1}\mu_{i+1}\ \bigg|\ l = \delta_\alpha(Y_{in,im+1}) + \beta,\ i < \operatorname{floor}\bigg(\frac{r-\beta}{m+n}\bigg)\bigg\} \\ &\cup \bigg\{Y_{in,im+1}\mu^{\mathbf{m}}\ \bigg|\ l = \delta_\alpha\big(Y_{in,im+1}\mu^{\mathbf{m}}\big), \operatorname{floor}\bigg(\frac{r-\beta}{m+n}\bigg) < i < k\bigg\} \\ &\cup \big\{Y_{2kn,2km+1}\mu^{\mathbf{m}}\ \big|\ l = \delta_\alpha\big(Y_{2kn,2km+1}\mu^{\mathbf{m}}\big)\big\}\Bigg). \end{aligned}$$

For any $(k-1)(m+n)+\beta < r < 2k(m+n)+\beta$, we have $E_r^{*,*} = E^{*,*}_{(k-1)(m+n)+\beta}$ and $E_\infty^{*,*} = E^{*,*}_{2k(m+n)+\beta}$. Thus,

$$E_\infty^{l,-l+1} = \begin{cases} \{0\}, & \text{if } l > 2k(m+n)+\beta, \\ \operatorname{span}\{Y_{2kn,2km+1}\mu_{k+1}\}, & \text{if } l = 2k(m+n)+\beta. \end{cases}$$

This completes the proof. $\square$

Remark 6.4. An alternative parametric normal form to Eq. (6.5) is

$$\tilde{v}^{(2\beta)} = \begin{pmatrix} mx \\ -bx^{2kn}y^{2km+1} + ax^{kn}y^{km+1} + \sum_{i=0}^{k} x^{in}y^{im+1}\mu_{i+1} + ny \end{pmatrix}, \tag{6.6}$$

but this requires another approach and is beyond the scope of this paper.

7. Parametric generalized Hopf singularity

In this section, we present the parametric normal form of generalized Hopf singularity via the spectral sequence method, see also [1,6,8,25,38]. The following presentation of our algebraic structures was suggested in part by Murdock [21], see also [13,14,25]. We begin with the most general C^∞ system in two dimensions with vector parameter μ having a Hopf singularity at the origin. When expanded in a formal power series such a system takes on the form (modulo flat functions)

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix} + \sum \begin{pmatrix} a_{jl\mathbf{n}} \\ b_{jl\mathbf{n}} \end{pmatrix} x^j y^l \mu^{\mathbf{n}} \tag{7.1}$$

where summation is taken over $j, l \in \mathbb{N}_0$, $\mathbf{n} \in \mathbb{N}_0^q$, $j+l+|\mathbf{n}| > 1$, and $j+l \geqslant 1$. Introducing the complex variable $z = x + iy$ we obtain

$$\dot{z} = iz + \sum (b_{jl\mathbf{n}} + ia_{jl\mathbf{n}}) \left(\frac{z-\bar{z}}{2i}\right)^j \left(\frac{z+i\bar{z}}{2}\right)^l \mu^{\mathbf{n}}$$

which can be expanded in the form

$$\dot{z} = iz + \sum A_{jl\mathbf{n}} z^j \bar{z}^l \mu^{\mathbf{n}} \tag{7.2}$$

with $A_{jl\mathbf{n}} \in \mathbb{C}$. We now consider the following system, defined on $\mathbb{C}^2$, with variables (z, w):

$$\begin{pmatrix} \dot{z} \\ \dot{w} \end{pmatrix} = i \begin{pmatrix} z \\ -w \end{pmatrix} + \sum \begin{pmatrix} A_{jl\mathbf{m}} \\ B_{jl\mathbf{m}} \end{pmatrix} z^j w^l \mu^{\mathbf{m}}, \tag{7.3}$$

where $\overline{B_{jl\mathbf{m}}} = A_{jl\mathbf{m}}$. Eq. (7.3) reproduces our original system on the reality subspace of $\mathbb{C}^2$ defined by $w = \bar{z}$. (This is a real vector space, that is, it is a subspace of $\mathbb{C}^2$ over $\mathbb{R}$. For a complete discussion of reality conditions in normal form theory, see [21, pp. 203–206].) If we now set

$$X_{jl} = \begin{pmatrix} z^j w^l \\ w^j z^l \end{pmatrix}, \qquad Y_{jl} = \begin{pmatrix} iz^j w^l \\ iw^j z^l \end{pmatrix},$$

and write $A_{jl\mathbf{m}} = \overline{B}_{jl\mathbf{m}}$ with α and β real, our system on $\mathbb{C}^2$ takes the simple form

$$\begin{pmatrix} \dot{z} \\ \dot{w} \end{pmatrix} = Y_{10} + \sum \alpha_{jl\mathbf{m}} X_{jl} \mu^{\mathbf{m}} + \sum \beta_{jl\mathbf{m}} Y_{jl} \mu^{\mathbf{m}}, \tag{7.4}$$

where the reality conditions are now simply given by the α and β coefficients lying in $\mathbb{R}$. It is easy to see that the case when the original system is complex (that is, $(x, y) \in \mathbb{C}^2$) is included in the above formula. We simply take the field $\mathbb{F}$ to be either $\mathbb{R}$ or $\mathbb{C}$, and consider the right-hand side of Eq. (7.2) to be a vector field on $\mathbb{C}^2$ with coefficients in $\mathbb{F}$. From here on we take (7.4) as the starting form for our analysis. This way we avoid the formulas for the coefficients of (7.4) in terms of those of (7.1), which are of course rather complicated. Let

$$\mathscr{L} = \left\{ aY_{10} + \sum \alpha_{jl\mathbf{m}} X_{jl}\mu^{\mathbf{m}} + \sum \beta_{jl\mathbf{m}} Y_{jl}\mu^{\mathbf{m}} \;\middle|\; a, \alpha_{jl\mathbf{m}}, \beta_{jl\mathbf{m}} \in \mathbb{F},\ \mathbf{m} \in \mathbb{N}_0^q \right\}.$$

We also use the subalgebra

$$\mathscr{C} = \left\{ aY_{10} + \sum \alpha_{j+1,j\mathbf{m}} X_{j+1,j}\mu^{\mathbf{m}} + \sum \beta_{j+1,j\mathbf{m}} Y_{j+1,j}\mu^{\mathbf{m}} \right\},$$

which is a result of normalization in the classical sense (without time rescaling and reparametrization and prior to hypernormalization). Let the ordering of formal basis

$$\mathscr{B} = \left\{ X_{ij}\mu^{\mathbf{n}}, Y_{ij}\mu^{\mathbf{n}} \;\middle|\; \mathbf{n} \in \mathbb{N}_0^m,\ i, j \in \mathbb{N}_0,\ i + j > 0 \right\}$$

follow the rules (I) and (II), and Y-terms precede X-terms when they have the same grade. The reason behind the latter condition is that $\{Y_{i+1,i}, X_{i+1,i}\}$-terms only appear in classical normal forms (and in the higher level normal forms) and $Y_{i+1,i}$-terms represent phase terms while $X_{i+1,i}$-terms represent amplitude terms.

Time rescaling is defined by

$$t = \tau\big(1 + T(z\bar{z}, \mu)\big) = \tau + \tau \sum T_{l,\mathbf{m}} z^l \bar{z}^l \mu^{\mathbf{m}}, \tag{7.5}$$

where the summation is taken over $l \in \mathbb{N}_0$ and $\mathbf{m} \in \mathbb{N}_0^q$, only if $l + |\mathbf{m}| \geqslant 1$. Let $Z_l = z^l \bar{z}^l$ (in particular $Z_0 = 1$) and define

$$\mathscr{T} = \operatorname{span}\left\{ \sum T_{l,\mathbf{m}} Z_l \mu^{\mathbf{m}} \;\middle|\; l + |\mathbf{m}| \geqslant 1 \right\}. \tag{7.6}$$

Let $\{Z_i\mu^{\mathbf{n}}\}$ be a formal basis with an order satisfying the rules (I) and (II), when the grading function is defined by

$$\delta_\alpha\big(Z_i\mu^{\mathbf{m}}\big) = 2i + \alpha \cdot \mathbf{m}, \quad i \in \mathbb{N}_0.$$

Further, for any $Z_i\mu^{\mathbf{n}_1} \in \{Z_i\mu^{\mathbf{n}}\}$ and $X_{jl}\mu^{\mathbf{n}_2}, Y_{jl}\mu^{\mathbf{n}_2} \in \mathscr{B}$, we have

$$Z_i\mu^{\mathbf{n}_1} X_{jl}\mu^{\mathbf{n}_2} = X_{(i+j)(i+l)}\mu^{\mathbf{n}_1+\mathbf{n}_2} \quad \text{and} \quad Z_i\mu^{\mathbf{n}_1} Y_{jl}\mu^{\mathbf{n}_2} = Y_{(i+j)(i+l)}\mu^{\mathbf{n}_1+\mathbf{n}_2}.$$

The following lemma presents the first level parametric normal form of system (7.1).

Lemma 7.1. *For any $\alpha \in \mathbb{N}^q$, there exists $(S, \mathbf{0}, T) \in \bigoplus_{n=1}^{\infty} A^{0,n}$ that transforms v, given by*

$$v = Y_{10} + \sum_{i+j+r=2,\ |\mathbf{n}|=r,\ i+j\geqslant 1}^{\infty} a_{ij\mathbf{n}}^{(0)} X_{ij}\mu^{\mathbf{n}} + \sum_{i+j+r=2,\ |\mathbf{n}|=r,\ i+j\geqslant 1}^{\infty} b_{ij\mathbf{n}}^{(0)} Y_{ij}\mu^{\mathbf{n}}, \tag{7.7}$$

into the first level extended partial parametric normal form

$$\tilde{v}^{(1)} = \Phi_S \circ \Theta_T(v) \in Y_{10} + \bigoplus_{n=1}^{\infty} \mathscr{N}_n^{(1)}, \tag{7.8}$$

where

$$\mathscr{N}_n^{(1)} = \operatorname{span}\{X_{(l+1)l}\mu^{\mathbf{n}} \mid n = 2l + \alpha \cdot \mathbf{n}, \mathbf{n} \in \mathbb{N}_0^q\} \subseteq \mathscr{C}_n$$

for $n \in \mathbb{N}$. *Furthermore, we have* $\operatorname{Total}^0(E_1^{*,*}) = \bigoplus_{n=0}^{\infty} \{0\} \times \mathscr{P}_n \times \mathscr{C}_n$.

Proof. For any $n \in \mathbb{N}$, we have

$$Z_1^{n,-n+1} = \mathscr{F}^n \mathscr{L}, \qquad Z_0^{n+1,-n} = \mathscr{F}^{n+1}\mathscr{L} \quad \text{and} \quad B_0^{n,-n+1} = \mathscr{F}^n\mathscr{L} \cap d_0(\mathscr{F}^n A^{0,*}),$$

where $d_v(S, P, T) = D_\mu v P + T v + \operatorname{ad}_S v$. Therefore,

$$\begin{aligned} B_0^{n,-n+1} + Z_0^{n+1,-n} &= D_\mu Y_{10}\mathscr{P}_n + \mathscr{R}_n Y_{10} + \operatorname{ad}_{Y_{10}}\mathscr{L}_n + \mathscr{F}^{n+1}\mathscr{L} \\ &= \mathscr{R}_n Y_{10} + \operatorname{ad}_{Y_{10}}\mathscr{L}_n + \mathscr{F}^{n+1}\mathscr{L} \\ &= \operatorname{span}\{Y_{(l+1)l}\mu^{\mathbf{n}} \mid n = 2l + \alpha \cdot \mathbf{n},\ l \in \mathbb{N}_0\} + \mathscr{C}_n^c + \mathscr{F}^{n+1}\mathscr{L} = \mathscr{T}_n. \end{aligned}$$

Thus, by Lemma 4.1 we have $E_1^{n,-n+1} = \mathscr{F}^n\mathscr{L}/\mathscr{T}_n^{(1)} = (\mathscr{N}_n^{(1)} + \mathscr{T}_n^{(1)})/\mathscr{T}_n^{(1)}$, where

$$\mathscr{N}_n^{(1)} = \operatorname{span}\{X_{(l+1)l}\mu^{\mathbf{n}} \mid n = 2l + \alpha \cdot \mathbf{n},\ l \in \mathbb{N}_0\}.$$

Since

$$Z_1^{n,-n} = \{0\} \times \mathscr{P}_n \times \mathscr{C}_n + \mathscr{F}^{n+1}A^{0,*}, \qquad Z_0^{n+1,-n-1} = \mathscr{F}^{n+1}A^{0,*}$$

and $B_0^{n,-n} = \{\mathbf{0}\}$, we have $E_1^{n,-n} = \{0\} \times \mathscr{P}_n \times \mathscr{C}_n + \mathscr{F}^{n+1}A^{0,*}/\mathscr{F}^{n+1}A^{0,*}$. Then,

$$\operatorname{Total}^1(E_1^{*,*}) = \bigoplus_{r=0}^{\infty} E_1^{r,-r+1} = \bigoplus_{r=0}^{\infty} \mathscr{N}_r^{(1)} \subset \mathscr{L}.$$

Besides, $\operatorname{Total}^0(E_1^{*,*}) = \bigoplus_{n=0}^{\infty} E_1^{n,-n} = \bigoplus_{n=0}^{\infty}\{0\} \times \mathscr{P}_n \times \mathscr{C}_n$.

The rest of the proof is straightforward following Lemma 4.2. $\square$

This lemma implies that the first level partial extended parametric normal form is

$$\tilde{v}^{(1)} = Y_{10} + \sum a_{i,\mathbf{n}}^{(1)} X_{i+1,i}\mu^{\mathbf{n}}, \tag{7.9}$$

where the summation is over $i \geqslant 0$ and $|\mathbf{n}| \geqslant 0$. Following [11, p. 385], we say that v is a *Hopf singularity of order* k, when

$$k := \min\{i \mid a_{i,\mathbf{0}}^{(1)} \neq 0\}. \tag{7.10}$$

In this case, the origin is a weak focus of order k for $v(x, \mu = \mathbf{0})$. Without loss of generality, we may assume that $a_{k,\mathbf{0}}^{(1)} = 1$. Systems with $k > 1$ are called *generalized Hopf singularities.* Now we intend to put a condition on parametric terms of the form $X_{i+1,i}\mu_j$. So, define a $(k+1) \times q$ matrix

$$B^{(1)} = (b_{ij}) \quad \big(b_{ij} = a_{i\mathbf{e}_j}^{(1)} \text{ for } i \leqslant k, \text{ and } b_{k+1,j} = a_{2k\mathbf{e}_j}^{(1)}\big). \tag{7.11}$$

When $\operatorname{rank} B^{(1)} = k + 1$, we refer to v as a *non-degenerate deformation* for the vector field $v^{(1)}(x, \mathbf{0})$.

Now it is time to use a linear reparametrization. Similar to the previous section, we apply the linear reparametrization on vector fields rather than on $E_1^{*,*}$-space. This simplifies non-degenerate deformation of first level normal form vector fields and then, we restart a new spectral sequence by defining $E_2^{*,*}$-space. This clearly violates the main property of spectral sequences, that is, each level is the cohomology of previous level. However, the most important point is that this procedure does not violate the rules of normal form computations. The later is the main purpose of this paper.

The condition (7.11) merely means that the parameters of the system are enough and in the right places to make $B^{(1)}$ an onto matrix. Thus, there exists an invertible linear reparametrization $\nu = C\mu$ such that $\nu_i = \pi_i(B^{(1)}C\mu)$ for $i \leqslant k$. Therefore, the first level extended partial parametric normal form (7.9) can be sent to (denoting again μ instead of ν) $\tilde{v}^{(2)} = \Psi_C(\tilde{v}^{(1)})$ given by

$$\tilde{v}^{(2)} = Y_{10} + X_{k+1,k} + \sum_{i=0}^{k-1} X_{i+1,i}\mu_{i+1} + X_{2k+1,2k}\mu_{k+1} + \sum_{i=k+1}^{\infty} a_{i,\mathbf{n}}^{(2)} X_{i+1,i} + \sum_{i=k,\ i\neq 2k}^{\infty} a_{i,\mathbf{n}}^{(2)} X_{i+1,i}\mu^{\mathbf{n}} + \sum_{i=0}^{\infty} a_{i\mathbf{n}}^{(2)} X_{i+1,i}\mu^{\mathbf{n}}, \tag{7.12}$$

where the last two summations are taken over $|\mathbf{n}| = 1$ and $|\mathbf{n}| > 1$, respectively. Further, $a_{i,\mathbf{0}}^{(2)} = a_{i,\mathbf{0}}^{(1)}$. We choose the weights for parameters such that the terms in the form of $X_{i+1,i}\mu_{i+1}$ would have the same grade; that is, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{k+1})$,

$$\alpha_i = 4k - 2i + 3 \quad (i \leqslant k), \quad \text{and} \quad \alpha_{k+1} = 1. \tag{7.13}$$

Thus, $\delta_\alpha(X_{i+1,i}\mu^{\mathbf{n}}) = 2i + \alpha \cdot \mathbf{n}$, in particular we have $\delta_\alpha(X_{i+1,i}\mu_{i+1}) = \delta_\alpha(X_{2k+1,2k}\mu_{k+1}) = 4k+1$ for $i \leqslant k$. This will significantly simplify our work compared with what is presented in [13].

Define

$$\text{Total}^1\big(E_2^{*,*}\big) := \bigoplus_{r=0}^{\infty} E_2^{r,-r+1}, \tag{7.14}$$

where $E_2^{0,1} = \operatorname{span}\{Y_{10}\}$; $E_2^{r,-r+1} = \{0\}$ if $0 < r < 2k$ or $2k+1 < r < 4k$;

$$E_2^{2k,-2k+1} = \operatorname{span}\{X_{k+1,k}\}; \qquad E_2^{4k,-4k+1} = \operatorname{span}\{X_{2k+1,2k}\};$$

and finally $E_2^{r,-r+1} = \operatorname{span}\{X_{i,i-1}\mu^{\mathbf{n}} \mid r = 2i - 1 + \alpha|\mathbf{n}|\}$ for any $r > 4k$. Roughly speaking $\text{Total}^1(E_2^{*,*})$ is the vector space span of all vector fields given by Eq. (7.12). Denote

$$\mathcal{V} := \operatorname{span}\left\{ X_{2k+1,2k}\mu_{k+1} + \sum_{i=1}^{k} X_{i,i-1}\mu_i \right\},$$

and note that $\mathcal{V} \subset E_2^{4k+1,-4k}$. Further, we define

$$\text{Total}^0\big(E_2^{*,*}\big) = \bigoplus_{n=1}^{\infty} E_2^{n,-n} = \bigoplus_{n=1}^{\infty} \{0\} \times \mathscr{P}_n \times \mathscr{C}_n.$$

It can be seen that the set of transformations generated by elements from $\text{Total}^0(E_2^{*,*})$ constructs a subgroup $\mathcal{G}$ from G.

Remark 7.2. The invertible linear reparametrization was done outside the spectral sequences procedure, and then we redefined $\mathrm{Total}^1(E_2^{*,*})$ in Eq. (7.14). This destroys our spectral sequence structure, i.e., $H(E_1^{*,*}) \neq E_2^{*,*}$. The reason to do this was: G does not act on $\mathscr{L}$ as an initially linear map and the linear part of the action does not have the sole contribution in the normal form computation. The problem only refers to the linear transformations. Therefore, we applied the linear reparametrization on the vector fields to formulate them in a certain form and then, we considered the more restricted space $\mathrm{Total}^1(E_2^{*,*})$ and used the remaining spectral data for further normalization. Therefore, as far as the spectral sequences are concerned, we may only discuss E_r-terms for $r \geqslant 2$. This can be addressed as follows. Let us denote

$$M := E_2^{0,1} \oplus E_2^{2k,-2k+1} \oplus \mathcal{V}$$

and consider the quotient space $\mathscr{N}_2 := \mathrm{Total}^2(E_2^{*,*})/M$ (note that $\mathcal{V}$ is a $\mathscr{B}$-space and thus in particular, $\mathrm{Total}^2(E_2^{*,*}) = \mathcal{V} \oplus \mathscr{N}_2$). Then, define the induced group action $\mathcal{G}$ on the quotient space $\mathscr{N}_2$ by

$$\left(gw = g\left(Y_{10} + X_{k+1,k} + X_{2k+1,2k}\mu_{k+1} + \sum_{i=1}^{k} X_{i,i-1}\mu_i + w\right) \bmod (M)\right)$$

for any $w \in \mathscr{N}_2$ and $g \in \mathcal{G}$. One can observe that this group action is initially linear with respect to the associated filtration with the quotient space when $q = k+1$.

Lemma 7.3. *Let $v^{(2)} \in \mathrm{Total}^1(E_2^{*,*})$ (given in Eq. (7.14)), α given by Eq. (7.13). Then, we have*

$$\tilde{v}^{(2k)} = \tilde{v}^{(2)}, \qquad \mathrm{Total}^1\big(E_{2k}^{*,*}\big) = \mathrm{Total}^1\big(E_2^{*,*}\big), \qquad \mathrm{Total}^0\big(E_{2k}^{*,*}\big) = \mathrm{Total}^0\big(E_2^{*,*}\big),$$

and $\tilde{v}^{(1)}$ can be transformed to the $(2k+1)$-th level partial extended normal form $\tilde{v}^{(2k+1)} \in \mathrm{Total}^1(E_{2k+1}^{*,*})$, where

$$\mathrm{Total}^1\big(E_{2k+1}^{*,*}\big) = M \oplus \mathrm{span}\big\{X_{(l+1)l}\mu^{\mathbf{n}},\ X_{2k+1,2k}\mu^{\mathbf{n}} \bigm| l < k,\ |\mathbf{n}| > 1\big\}. \tag{7.15}$$

Furthermore,

$$\mathrm{Total}^0\big(E_{2k+1}^{*,*}\big) = \{0\} \times \mathscr{P}_l \times \bigoplus \mathrm{span}\big\{Y_{l+1,l}\mu^{\mathbf{m}}, X_{k+1,k}\mu^{\mathbf{m}}\big\}.$$

Proof. The first part is straightforward. According to Lemma 7.1, for $r = 2k+1$ and $m > 2k$, we have $d^{-1}\mathscr{F}^{m+2k}A^1 \cap \mathscr{F}^m A^0 = \{0\} \times \mathscr{P}_m \times \mathscr{L}_{H,m} \bmod (\mathscr{F}^{m+1}A^0)$. Thus, the equation $[X_{l+1,l}, X_{k+1,k}] = 2(l-k)X_{(l+k+1)(l+k)}$ completes the proof for Eq. (7.15). Let state terms precede time terms in the formal basis costyle. Hence,

$$\mathrm{Total}^0 E_{2k+1}^{*,*} = \{0\} \times \mathscr{P} \times \bigoplus \mathrm{span}\big(\big\{Y_{(l+1)l}\mu^{\mathbf{m}}, X_{(k+1)k}\mu^{\mathbf{m}}\big\}\big).$$

This completes the proof. □

Theorem 7.4. *Assume that the hypotheses of Lemma 7.3 hold for v. Then, we have*

$$\mathrm{Total}^1\big(E_{4k+2}^{*,*}\big) = \mathrm{span}\left\{Y_{10}, X_{k+1,k}, X_{2k+1,2k}\mu_{k+1} + \sum_{i=1}^{k} X_{i,i-1}\mu_i, X_{2k+1,2k}\right\}. \tag{7.16}$$

Furthermore, by invertible transformations v *can be sent into its infinite level parametric normal form* $v^{(\infty)} = v^{(4k+2)}$, *i.e.,* $\mathrm{Total}^1(E_\infty^{*,*}) = \mathrm{Total}^1(E_{4k+2}^{*,*})$.

Proof. It is easy to observe from Eq. (7.11) that $B^{(4k+1)} := (b_{\mathbf{i}\mathbf{e}_j}^{(4k)}) = B^{(2)}$, and

$$\mathrm{Total}^0\big(E_{4k+1}^{*,*}\big) = \mathrm{Total}^0\big(E_{2k+1}^{*,*}\big) \quad \text{and} \quad \mathrm{Total}^1\big(E_{4k+1}^{*,*}\big) = \mathrm{Total}^1\big(E_{2k+1}^{*,*}\big).$$

Therefore, we have $\langle \mathscr{P}_n, \mathbf{e}_i\rangle = \mathscr{P}^1_{n+\alpha_i} := \mathrm{span}\{\mu^{\mathbf{m}} \mid |\mathbf{m}| = n+\alpha_i\}$, and

$$D_\mu\big(v_{4k+1}^{4k+1}, \mathscr{P}_n\big) = \bigoplus_{i=1}^{k}\big\{X_{i,i-1}\mathscr{P}^1_{n+\alpha_i}\big\} \oplus \big\{X_{2k+1,2k}\mathscr{P}^1_{n+\alpha_{k+1}}\big\}.$$

Thus, $D_\mu(v_{4k+1}^{4k+1}, \mathscr{P}_m) + \mathscr{T}_m^{(4k+1)} = \mathscr{L}_m$ for any $m > 4k+1$. Thereby,

$$\begin{aligned}
E_{4k+2}^{m,-m+1} &= \frac{\mathscr{F}^m A^1}{d(\mathscr{F}^{m-4k-1}A^0) \cap \mathscr{F}^m A^1 + \mathscr{T}_m^{(4k+1)} + \mathscr{F}^{m+1}A^1} \\
&= \frac{\mathscr{F}^n A^1}{d(\mathscr{F}^{m-4k-1}\mathrm{Total}^0(E_{2k+1}^{*,*})) \cap \mathscr{F}^m A^1 + \mathscr{T}_m^{(4k+1)} + \mathscr{F}^{m+1}A^1} \\
&= \frac{\mathscr{F}^m A^1}{d(\mathscr{F}^{m-4k-1}(\{0\} \times \mathscr{P} \times \{0\})) \cap \mathscr{F}^m A^1 + \mathscr{T}_m^{(4k+1)} + \mathscr{F}^{m+1}A^1} \\
&= \frac{\mathscr{F}^m \mathscr{L}}{D_\mu(v_{4k+1}^{4k+1}, \mathscr{P}_m) + \mathscr{T}_m^{(4k+1)} + \mathscr{F}^{m+1}\mathscr{L}} = \{0\},
\end{aligned}$$

where $m > 4k+1$. Therefore, the sequence $\{E_r^{*,*} \mid r \geqslant 2\}$ collapses at $4k+2$.

The proof is complete. □

Corollary 7.5. *Let* v *be a Hopf singularity of order* k, $k \in \mathbb{N}$, *and* v *be a non-degenerate deformation for* $v(x, \mathbf{0})$, *that is,* $\mathrm{rank}\, B^{(1)} = k+1$. *Then, the invertible transformation* $g_\infty \in G$, *constructed according to Theorem* 4.4, *transforms* v *into the infinite level parametric normal form*

$$v^{(\infty)} = Y_{10} + \sum_{l=1}^{k} X_{l(l-1)}\mu_l + X_{(k+1)k} + aX_{(2k+1)2k} + X_{(2k+1)2k}\mu_{k+1}, \tag{7.17}$$

where $a \in \mathbb{F}$, *or equivalently system* (7.1) *can be transformed into*

$$\frac{1}{\rho}\frac{d\rho}{dt} = a\rho^{4k} + \rho^{2k} + \rho^{4k}\mu_{k+1} + \sum_{l=1}^{k}\rho^{2l-2}\mu_l \quad \text{and} \quad \frac{d\theta}{dt} = 1. \tag{7.18}$$

Proof. Note that $g_\infty = \Lambda_C \circ \Phi_S \circ \Psi_P \circ \Theta_T$ for some $(S, P, T) \in \mathscr{T} \times \mathscr{P} \times \mathscr{S}$ and $C \in GL(q)$. The proof is straightforward following Theorems 7.4 and 4.4. □

Remark 7.6. One can modify our method to obtain an alternative parametric normal form to Eq. (7.18), given by

$$\frac{1}{\rho}\frac{d\rho}{dt} = \rho^{2k} + \sum_{l=1}^{k} \rho^{2l-2}\mu_l \quad \text{and} \quad \frac{d\theta}{dt} = 1 + \rho^{2k}\mu_{k+1} + a\rho^{2k}. \tag{7.19}$$

One may consider only k parameters in the system provided that they split the k *zero* roots (associated with $R = \rho^2$) for the unperturbed system ($\mu = \mathbf{0}$) of the amplitude equation (7.19). This system was called *parametric generic* in [14]. Its infinite level normal form has an infinitely many terms in the phase equation (7.19), see [14,38,39].

8. Conclusion

The method of spectral sequences has been suitably generalized to consider the infinite level normal forms of parametric vector fields. Our results can also be considered as a generalization of the spectral sequences of orbital normal forms of non-parametric systems. The method has been applied to three cases with multiple parameters: single zero, resonant saddle and generalized Hopf singularities.

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